INFINITE SERIES

BY

TOMLINSON FORT
PROFESSOR OF MATHEMATICS, LEHIGH UNIVERSITY

OXFORD
AT THE CLARENDON PRESS
1930

OXFORD UNIVERSITY PRESS
AMEN HOUSE E.C. 4
LONDON EDINBURGH GLASGOW
LEIPZIG NEW YORK TORONTO
MELBOURNE CAPETOWN BOMBAY
CALCUTTA MADRAS SHANGHAI
HUMPHREY MILFORD
PUBLISHER TO THE
UNIVERSITY

Printed in Great Britain

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CHAPTER I

THE NUMBER SYSTEM

§ 1. An understanding of the processes of this book can only follow an exact knowledge of the number system of ordinary algebra. A discussion of the number system in any completeness would consume so large an amount of space and require so much detailed study that it is thought inadvisable here. A knowledge of the rational numbers as an ordered set together with the fundamental operations on them will be assumed. The theory of infinite series is, however, so closely connected with the theory of irrational numbers, that in order for the following work to be logically complete, definitions and a short discussion of the irrational numbers are given following Dedekind.*

§ 2. The Irrational Numbers.

Begin with the rational numbers, that is the integers, zero, and the rational fractions. They are ordered. The notion of order is assumed as fundamental; and given any two rational numbers, N_1 and N_2, we know how to tell which is of the higher order. If N_2 is of higher order than N_1 we say that N_2 is larger or greater than N_1 or that N_1 is smaller or less than N_2.

Any particular one of the rational numbers, N, divides the entire set of rational numbers into two classes, c_1 and c_2, c_1 those which are smaller than N and c_2 those which are larger than N. N itself can be arbitrarily assigned to either class. Every number in c_1 is smaller than any number in c_2. We say that by means of N we have formed a section of the rational numbers. More generally—

Definition 1. *Whenever, by any method the rational numbers are divided into two classes, c_1 and c_2, such that c_1 con-*

* See, for example, R. Dedekind, *Stetigkeit und irrationale Zahlen*, or almost any book on 'Functions of a Real Variable'.

tains at least one rational number and c_2 at least one rational number and, such that any rational number whatever lies in either c_1 or c_2 and, such that every rational number in c_2 is greater than any rational number whatever in c_1, a section of the rational numbers is said to have been formed.

There are other ways to form sections than that explained above by means of the rational number N. It is to be noted that in the type of section generated that way, either c_1 has a largest rational number or c_2 a smallest, namely the number N itself. An example of a different type of section will now be given. There is no rational number which, when squared, yields 2.*

We let c_1 contain all rational numbers less than or equal to 0 and each rational number greater than 0 whose square is less than 2, c_2 all positive rational numbers whose squares are greater than 2. As positive rational numbers have the same order as their squares, we have formed a section. Moreover, given any rational number whatever, it is possible † to find, (a) another smaller rational number whose square is greater than 2 or, (b) another larger rational number whose square is less than 2. Consequently, c_1 has no largest rational number nor c_2 a smallest. This characterizes the section as being of a different character than those generated by means of the rational numbers themselves.

A new set of objects, the real numbers, is now defined.

Definition 2. *To every section (c_1, c_2) of the rational numbers generated in any manner whatever, there corresponds a real number.* The number itself can be thought of

* Assume $\left(\frac{p}{q}\right)^2 = 2$ where p and q are integers with no common factor other than 1. Then $p^2 = 2q^2$ or p^2 is an even integer. Hence, as no odd integer when squared yields an even integer, p is even. Call it $2k$. Then $q^2 = 2k^2$ and consequently q is even. In other words p and q have the factor 2 in common, which is a contradiction.

† If k is any positive rational number such that $k^2 < 2$ and if $l = \frac{4+3k}{3+2k}$, then $l > k$ but $l^2 < 2$ and if $k^2 > 2$, $l < k$ but $l^2 > 2$.

These conclusions are drawn from the relations $l - k = 2(2-k^2)/(3+2k)$ and $2 - l^2 = (2-k^2)/(3+2k)^2$.

as the section itself, as the class c_1 of the section, or more simply as the symbol that we write.

Definition 3. *In case the section is not made by a rational number, N, that is in case c_1 contains no largest rational number nor c_2 a smallest rational number, the real number is said to be irrational.*

Definition 4. *In case c_1 has a largest rational number or c_2 a smallest, the real number is said to be rational and to correspond to the rational number forming the section.* In the study of infinite series the two can be considered as identical * and will be so considered here.

Theorem 1. *The set of real numbers can be ordered.*

PROOF: (1) The rational numbers are already ordered.

(2) An irrational number formed by the section (c_1, c_2) is ordered with reference to the rational numbers as follows. It is larger (of higher order) than any rational number lying in c_1 and smaller (of lower order) than any rational number lying in c_2.

(3) Two irrational numbers formed by (c_1, c_2) and (c_1', c_2'), which we denote by C and C' respectively, are said to coincide if every rational number in c_1 lies in c_1' and every rational number in c_2 lies in c_2' and conversely. As a matter of fact, if the classes c_1 and c_1' contain identical elements, c_2 and c_2' necessarily do also and the numbers coincide.

(4) Assume the numbers C and C' different. A certain rational number which we call n lies in c_1 and not in c_1' or in c_1' and not in c_1. Assume the first. Then n necessarily lies in c_2'. Hence, by (2) C is greater than n and C' less. We agree that C' shall be less than C. A definite order is thus established between C and C'. Moreover, if a third number C'' is greater than C it is also greater than C'; because c_1'' will contain a rational number not in c_1 and hence not in c_1'.

* The rational real numbers are a sub-set of the real numbers. If a detailed discussion of the operations on the numbers were made here, it would be seen that in the sense of isomorphism, this sub-set is identical with the rational numbers.

B 2

A definite order is thus set up among the elements of the entire set of real numbers and is adopted.

Observe that just as any rational number makes a section of the rational numbers, so any real number makes a section of the real numbers. Moreover, there can be no other method of making a section of the real numbers; because any such section would necessarily be also a section of the rational numbers and hence by definition would define a real number.

The notation $>$, $=$, $<$, for greater than, equal to, and less than, or the combined symbols, $\leqq$, $\geqq$, have an ordinal interpretation. There is no reference to measurement, although they may be applied to problems in measurement, and problems in measurement may be introduced for purposes of illustration. Indeed, the real numbers can be referred to as points on a line or as line segments, and applications in geometry be made. It should be borne in mind, however, that the facts are arithmetic ones concerning the elements of a certain ordered set.

The operations of arithmetic should now be defined on the system of real numbers. These definitions are omitted. The fundamental laws which the operations obey are undoubtedly familiar to the reader. A discussion of the call for the definitions and a proof of the various laws, although necessary in a complete discussion of numbers, is not, like the nature of the irrational numbers, an essential introduction to infinite series. The facts of the laws of combination are the important thing. These are well known to every reader of these pages. A logical discussion is available in various books.

§ 3. The Complex Numbers.

The passage from the real to the complex number system is a simple process and is generally given fairly well in elementary books, where the most usual graphical representation by means of the Gauss plane is also explained. We shall use the notation, $|a|$, to denote the absolute value or modulus of a.

There are two inequalities which will play so important a role in the subsequent discussions that they will be proved here. Due to their simplicity and frequent occurrence, reference to this proof will generally not be given.

Theorem 2. HYPOTHESIS: *A and B are any complex numbers whatever.* CONCLUSION:

(1) $$|A+B| \leqq |A| + |B|,$$

(2) $$||A| - |B|| \leqq |A+B|.$$

To prove (1), let $A = a+bi$ and $B = c+di$ where a, b, c, d are real, and assume the contrary to the inequality; that is,

$$\sqrt{(a+c)^2+(b+d)^2} > \sqrt{a^2+b^2} + \sqrt{c^2+d^2}.$$

From this follows $0 > (ad-bc)^2$, which is absurd.

To prove the second, let

$$A'+B' = A \text{ and } B' = -B; \text{ then,}$$

$$|A'+B'| \leqq |A'| + |B'|; \text{ that is,}$$

$$|A| \leqq |A+B| + |B|, \text{ from which}$$

(3) $$|A| - |B| \leqq |A+B|.$$

And in an exactly similar manner,

(4) $$|B| - |A| \leqq |A+B|.$$

But $||A| - |B||$ equals either $|A| - |B|$ or $|B| - |A|$. Consequently, (2) follows from (3) and (4).

If A and B are marked as points in the Gauss complex plane these inequalities imply the well-known geometric theorem: One side of a plane triangle is longer than the difference of the other sides but shorter than their sum.

EXERCISES

1. Prove that there is no rational number which, when raised to the n-th power, n an integer other than 1, yields 2.

2. Generalize the theorem of Exercise 1 and prove your generalization.

3. Use your results in the last exercise to form sections of the rational numbers and thus define certain irrational real numbers.

4. Assuming the operations of addition, subtraction, multiplication, and division of rational numbers, give exact definitions for these operations on the real numbers.

5. Using your definitions in Exercise 4, prove addition of real numbers commutative.

6. State and prove other laws for real numbers.

7. If a and b are positive rational numbers and k is a positive integer, after making necessary definitions, prove $\sqrt[k]{a} \gtreqless \sqrt[k]{b}$ according as $a \gtreqless b$.

8. Show how any process that yields an unending decimal fraction can be used to form a section of the rational numbers, and hence to define a real number.

9. Assuming the real numbers, explain a method for the formation of the complex number system.

10. Define the usual operations of arithmetic on the complex numbers, and prove the commutative law of multiplication.

11. State and prove other laws for operating with complex numbers.

12. Prove:

$$|a| = |-a|; \quad |a-b| = |b-a|;$$
$$|ab| = |a| \cdot |b|; \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|},\ b \neq 0;$$
$$|a_1 \cdot a_2 \cdot \ldots \cdot a_n| = |a_1| \cdot |a_2| \cdot \ldots \cdot |a_n|.$$

13. Prove:

If $|a-b| < A$ and $|b-c| < B$, then $|a-c| < A+B$.

14. Prove:

$$[\,|a+b|+|a-b|\,]^2 \geqq 2[\,|a|^2+|b|^2].$$

CHAPTER II

SEQUENCES

§1. Sequences and Series.

Definition 5. *We say that we have a simple infinite sequence of numbers, or briefly an infinite sequence; if when any particular positive integer, n, is given, there exist certain prescribed rules or conditions by which a particular number a_n is determined.*

It is not essential that the actual determination of a_n be a practical process or even a possible one; but simply that when the integer is given, we are assured that there corresponds a unique number. Nor is it essential that the numbers, n, be the positive integers. They may be any set of numbers in one to one correspondence with the positive integers.

Definition 6. *The numbers* (1) $a_1, a_2, a_3, \ldots,$ *where the dotted line means continued indefinitely, that is as far as desired, is called an infinite series.*

The distinction between series and sequences is not always sharply drawn. The distinction as here drawn and which will be adhered to, is largely one of point of view. When the word sequence is used the attention is fixed on one of the numbers at a time; whereas when the term series is used the attention is fixed on the set of numbers rather than on particular ones. Thus *a number a_n is called a term of the series* (1). The terms of the series constitute a sequence.

Definition 7. *A sequence, s_n, is said to approach a number s as a limit when n becomes infinite; if, given any particular positive number whatever, ϵ, it is possible to find a positive number, M, such that, when $n \geqq M$, $|s_n - s| < \epsilon$.*

We shall denote this by $\lim\limits_{n \to \infty} s_n = s$, or simply by $s_n \to s$.

It is to be noticed that no restriction is placed on ϵ other than that it be positive. The interesting fact, however, is that it can be made as small as desired.

§ 2. Fundamental theorems on sequences.

Theorem 3. HYPOTHESIS: *$s_n \to s_1$ and $s_n \to s_2$.* CONCLUSION: $s_1 = s_2$.

PROOF: Suppose the theorem not true and let

$$(1) \qquad |s_1 - s_2| = a > 0.$$

Let a fixed number ϵ be chosen such that $0 < \epsilon < a$. From Definition 7 it is possible to find an $M > 0$ and an $M' \geqq M$, such that when $n \geqq M$,

$$|s_1 - s_n| < \frac{\epsilon}{2}, \text{ and when } n \geqq M',$$

$$|s_2 - s_n| < \frac{\epsilon}{2}. \quad \text{But,}$$

$$s_1 - s_2 = (s_1 - s_n) - (s_2 - s_n). \quad \text{Hence,}$$

$$(2) \qquad |s_1 - s_2| \leqq |s_1 - s_n| + |s_2 - s_n| < \epsilon \text{ when } n \geqq M'.$$

(1) and (2) are impossible simultaneously. Hence, $s_1 = s_2$.

A different wording of this theorem is: It is impossible for a sequence to have two distinct limits.

Theorem 4. HYPOTHESIS: *$s_n = s$, a fixed number.* CONCLUSION: $s_n \to s$.

PROOF: Let any $\epsilon > 0$ be given, then $|s_n - s| = 0 < \epsilon$, which constitutes proof.

Theorem 5. HYPOTHESES: (i) $s_n \to s$; (ii) *c is constant.* CONCLUSION: $cs_n \to cs$.

PROOF: Assume $c \neq 0$, and let $\epsilon > 0$ be given. Choose M so large that when $n > M$, $|s_n - s| < \dfrac{\epsilon}{|c|}$. Then,

$$|cs_n - cs| = |c|\,|s_n - s| < \epsilon;$$

which constitutes proof. In case $c = 0$ this theorem falls under the preceding.

Theorem 6. HYPOTHESIS: $s_n \to s$. CONCLUSION: *It is possible to find a fixed number G such that $|s_n| < G$ always.*

PROOF: Choose a particular $\epsilon > 0$. Let M be a corresponding integer such that

$$(3) \qquad |s_n - s| < \epsilon \text{ when } n > M.$$

The numbers $|s_1|, \ldots, |s_M|$ are a definite set of M fixed numbers. Let a number larger than any of them be denoted by S. Now let G be the larger of the two numbers S and $|s| + \epsilon$. If $n \leqq M$, $|s_n| < S \leqq G$. If $n > M$, by (3),

$$|s_n| < |s| + \epsilon \leqq G.$$

Hence, this G satisfies the conditions of the theorem. Having found one such number, we can take any larger number and it will satisfy the conditions of the theorem also.

Theorem 7. HYPOTHESES: (i) $s_n \to 0$; (ii) $s_n = u_n + v_n i$, *where u_n and v_n are real.* CONCLUSION: *$u_n \to 0$ and $v_n \to 0$.*

PROOF: Given any $\epsilon > 0$, choose M so that $|s_n| < \epsilon$ when $n > M$. But $|u_n| \leqq |s_n|$ and $|v_n| \leqq |s_n|$. Hence $|u_n| < \epsilon$ and $|v_n| < \epsilon$ when $n > M$.

Corollary 1. HYPOTHESIS: $s_n = u_n + v_n i \to s = u + iv$. CONCLUSION: *$u_n \to u$ and $v_n \to v$.*

Proof is omitted.

Corollary 2. HYPOTHESES: (i) $s_n \to s$; (ii) *s_n is real.* CONCLUSION: *s is real.*

PROOF: $v_n = 0$ and hence, by Theorem 4, $v_n \to 0$. Hence, $v = 0$.

Theorem 8. HYPOTHESES: (i) *s_n and t_n are real*; (ii) $s_n \to s$; (iii) $t_n \to t$; (iv) *When $n \geqq k$, $t_n \leqq s_n$.* CONCLUSION: $t \leqq s$.

PROOF: Suppose $t - s = \delta > 0$. Choose $M > k$ so that when $n > M$,

$$|t_n - t| < \frac{\delta}{2} \text{ and } |s_n - s| < \frac{\delta}{2} \text{ simultaneously.}$$

$$t - s = t - t_n + s_n - s + t_n - s_n.$$

From which

$$t_n - s_n = \delta - [(t - t_n) + (s_n - s)].$$

From which, when $n > M$,

$$t_n - s_n > \delta - \left[\frac{\delta}{2} + \frac{\delta}{2}\right] = 0, \text{ a contradiction.}$$

Corollary 1. HYPOTHESES: (i) $s_n \to s$; (ii) $s_n \leqq c$. CONCLUSION: $s \leqq c$.

Corollary 2. HYPOTHESES: (i) $s_n \to s$; (ii) $s_n \geqq c$. CONCLUSION: $s \geqq c$.

Theorem 9. HYPOTHESES: $v_n \to 0$, $u_n \to 0$. CONCLUSION: $s_n = u_n + iv_n \to 0$.

PROOF: Let $\epsilon > 0$ be given, and let M be so large that when

$$n > M, \text{ both } |u_n| < \frac{\epsilon}{\sqrt{2}} \text{ and } |v_n| < \frac{\epsilon}{\sqrt{2}},$$

then
$$|s_n| = \sqrt{u_n^2 + v_n^2} < \epsilon.$$

Corollary. HYPOTHESES: $u_n \to u$, $v_n \to v$. CONCLUSION: $s_n \to u + vi$.

Theorem 10. HYPOTHESIS: $s_n \to s$. CONCLUSION: $|s_n| \to |s|$.

PROOF: Let $\epsilon > 0$ be given. Choose M so that when $n \geqq M$, $|s_n - s| < \epsilon$. Then $||s_n| - |s|| \leqq |s_n - s| < \epsilon$, and proof is complete.

Theorem 11. HYPOTHESES: $s_n \to s$, $\sigma_n \to \sigma$. CONCLUSION: $s_n + \sigma_n \to s + \sigma$.

PROOF: Given any $\epsilon > 0$, choose M so that when $n \geqq M$,

$$|s_n - s| < \frac{\epsilon}{2} \text{ and } |\sigma_n - \sigma| < \frac{\epsilon}{2}$$

also. Then,

$$|s_n + \sigma_n - s - \sigma| \leqq |s_n - s| + |\sigma_n - \sigma| < \epsilon;$$

which constitutes proof.

Corollary. HYPOTHESES:

$$s_n^{(1)} \to s^{(1)},\ s_n^{(2)} \to s^{(2)},\ \ldots,\ s_n^{(j)} \to s^{(j)}.$$

CONCLUSION:

$$s_n^{(1)} + s_n^{(2)} + \ldots + s_n^{(j)} \to s^{(1)} + s^{(2)} + \ \ldots\ + s^{(j)}.$$

PROOF is best made by means of mathematical induction from the theorem. Details are omitted.

Theorem 12. HYPOTHESES: $s_n \to s$, $\sigma_n \to \sigma$. CONCLUSION: $s_n - \sigma_n \to s - \sigma$.

PROOF: Let $\sigma_n = -\bar{\sigma}_n$ and $\sigma = -\bar{\sigma}$, and apply Theorems 5 and 11.

Theorem 13. HYPOTHESES: $s_n \to s$, $\sigma_n \to \sigma$. CONCLUSION: $s_n \sigma_n \to s\sigma$.

PROOF: Let $\epsilon > 0$ be given.

$$s\sigma - s_n\sigma_n = s(\sigma - \sigma_n) + \sigma_n(s - s_n).$$

$$(4) \qquad |s\sigma - s_n\sigma_n| \leqq |s||\sigma - \sigma_n| + |\sigma_n||s - s_n|.$$

Suppose $|s| < G$ and $|\sigma_n| < G$ (see Theorem 6), then choose M so that when $n > M$, $|\sigma - \sigma_n| < \dfrac{\epsilon}{2G}$ and $|s - s_n| < \dfrac{\epsilon}{2G}$ simultaneously.

It follows from (4) that $|s\sigma - s_n\sigma_n| < \epsilon$, completing the proof.

Theorem 14. HYPOTHESES: $s_n \to s$, $\sigma_n \neq 0$, $\sigma_n \to \sigma \neq 0$. CONCLUSION: $\dfrac{s_n}{\sigma_n} \to \dfrac{s}{\sigma}$.

PROOF: Let $\epsilon > 0$ be given, and suppose G a number such than $G > s$ and $G > \sigma$.

$$\frac{s_n}{\sigma_n} - \frac{s}{\sigma} = \frac{\sigma s_n - s\sigma_n}{\sigma_n \sigma} = \frac{\sigma(s_n - s) + s(\sigma - \sigma_n)}{\sigma_n \sigma}.$$

Choose M so that when $n \geqq M$,

$$|\sigma_n - \sigma| < \frac{\epsilon|\sigma|^2}{4G}, \quad |s_n - s| < \frac{\epsilon|\sigma|^2}{4G}, \quad |\sigma_n - \sigma| < \frac{|\sigma|}{2},$$

simultaneously. Then

$$\left|\frac{s_n}{\sigma_n} - \frac{s}{\sigma}\right| < \frac{\epsilon|\sigma|^2}{2|\sigma\sigma_n|} < \epsilon\frac{|\sigma|}{2|\sigma_n|} < \epsilon$$

and proof is complete.

Theorem 15. HYPOTHESIS: $s_n \to s$. CONCLUSION: *Given any $\epsilon > 0$, there exists an M such that when $n \geqq M$ and $p \geqq 0$ simultaneously,* $|s_n - s_{n+p}| < \epsilon$.

PROOF: $s_n - s_{n+p} = s_n - s + s - s_{n+p}$. Choose M so that when $n \geqq M$, $|s_n - s| < \dfrac{\epsilon}{2}$. Then

$$|s_n - s_{n+p}| \leqq |s_n - s| + |s - s_{n+p}| < \epsilon.$$

Theorem 16. HYPOTHESES: *Given a sequence s_n, and that when any $\epsilon > 0$ is given, it is possible to choose a fixed $M > 0$ such that*

$$|s_{n+p} - s_n| < \epsilon,$$

whenever $n \geqq M$ *and* $p \geqq 0$ *simultaneously.* CONCLUSION: s_n *approaches a limit*, s.

PROOF: First assume s_n real. This will include the case that s_n ultimately becomes and remains real. Choose a sequence of real numbers, ϵ_m, satisfying the conditions

(5) $0 < \epsilon_{m+1} < \epsilon_m$, (6) $\epsilon_m \longrightarrow 0$.

Let M_m be the least positive integer such that, whenever $n \geqq M_m$ and $p \geqq 0$ simultaneously,

(7) $$|s_{n+p} - s_n| < \epsilon_m.$$

Now, let N be a particular positive integer and let ν be an integer greater than N. Denote by a_ν the largest number among the numbers $s_{M_{N+1}} - \epsilon_{N+1}, \ldots, s_{M_\nu} - \epsilon_\nu$, and by b_ν the smallest number among the numbers $s_{M_{N+1}} + \epsilon_{N+1}, \ldots, s_{M_\nu} + \epsilon_\nu$. The inequality (7) can be written,

$$s_n - \epsilon_m < s_{n+p} < s_n + \epsilon_m.$$

Hence, when $n \geqq M_m$ and $\mu \geqq n$,

$$s_n - \epsilon_m < s_\mu < s_n + \epsilon_m,$$

and in particular,

(8) $$s_{M_m} - \epsilon_m < s_\mu < s_{M_m} + \epsilon_m.$$

Assume $\mu \geqq M_\nu$. If we were to write down the inequalities (8) from $m = N+1$ to $m = \nu$, among the first members we would find a_ν and among the third members b_ν. Consequently,

(9) $$a_\nu < s_\mu < b_\nu.$$

But as ν increases, N remaining fixed, a_ν certainly does not decrease nor b_ν increase, that is $a_\nu \leqq a_{\nu+1} < b_{\nu+1} \leqq b_\nu$. But $b_\nu \leqq s_{M_\nu} + \epsilon_\nu$ and $a_\nu \geqq s_{M_\nu} - \epsilon_\nu$. Hence,

(10) $$b_\nu - a_\nu \leqq (s_{M_\nu} + \epsilon_\nu) - (s_{M_\nu} - \epsilon_\nu) = 2\epsilon_\nu,$$

which approaches zero as $\nu \longrightarrow \infty$. Then, either there is a number s, such that

(11) $$a_\nu \leqq s \leqq b_\nu$$

for all values of ν; or if we consider any real number, c, whatever, for all sufficiently great values of ν, either $a_\nu > c$ or $b_\nu < c$. Whereupon we have a section of the real numbers which section defines a number s and

(11) $$a_\nu \leqq s \leqq b_\nu.$$

As a consequence of (9), (10), and (11),

(12) $$|s_\mu - s| < b_\nu - a_\nu \leqq 2\epsilon_\nu.$$

But the only condition on μ is $\mu > M_\nu$, and $\epsilon_\nu \longrightarrow 0$. Consequently, given any $\epsilon\,(\epsilon_\nu)$ we can find an $M\,(M_\nu)$ such that when $n \geqq M$, $|s_n - s| < \epsilon$, which is what we wished to prove.

The case where $s_n = u_n + v_n i$, u_n and v_n real, follows as a corollary. For if $|s_{n+p} - s_n| < \epsilon$, then $|u_{n+p} - u_n| < \epsilon$ and $|v_{n+p} - v_n| < \epsilon$; and hence, u_n and v_n approach limits; and hence, by theorem 9, s_n does also.

Corollary 1. *The inequality*, $|s_{n+p} - s_n| < \epsilon$, *of the theorem can be replaced by* $|s_{n'} - s_n| < \epsilon$, *where* $n' \geqq M$ *and* $n \geqq M$.

Corollary 2. *The inequality*, $|s_{n+p} - s_n| < \epsilon$, *of the theorem can be replaced by* $|s_n - s_M| < \epsilon$.

Similar corollaries might have been given for the previous theorem.

It is hardly necessary to remark that in the statement of the theorem, as in many other places, the symbol $<$ can be replaced by $\leqq$ and $\geqq$ by $>$.

Theorem 17. HYPOTHESES: (i) s_n *real*; (ii) $s_n < G$, *a fixed number*; (iii) $s_{n+1} \geqq s_n$ *when* $n > M$, *a fixed number.* CONCLUSION: *There exists a real number*, s, *such that* $s_n \longrightarrow s$ *and* $s \leqq G$.

PROOF: Suppose that s_n does not approach a limit. There exists, then, a fixed $\epsilon > 0$, such that it is impossible to find an m such that

(13) $|s_{n+p} - s_n| < \epsilon$, when $n > m$ and $p \geqq 0$.

Choose such an ϵ and then denote a particular fixed $m > M$ by m_1 and particular values of $n > m_1$ and $p > 0$ for which (13) is not true by n_1 and p_1. Then,

$$s_{n_1+p_1} - s_{n_1} \geqq \epsilon.$$

Now choose a second $m > n_1 + p_1$ and denote it by m_2, and

denote an $n > m_2$ and $p > 0$ for which (13) does not hold by n_2 and p_2. Then,

$$s_{n_2+p_2} - s_{n_2} \geqq \epsilon.$$

Now choose a third value $m_3 > n_2 + p_2$ and so on. Adopting the obvious notation, we have

$$s_{n_k+p_k} - s_{n_k} \geqq \epsilon.$$

Adding all these inequalities,

$$s_{n_k+p_k} - (s_{n_k} - s_{n_{k-1}+p_{k-1}}) - \ldots - (s_{n_2} - s_{n_1+p_1}) - s_{n_1} \geqq k\epsilon.$$

Dropping that which is surely negative or zero,

$$s_{n_k+p_k} - s_{n_1} \geqq k\epsilon.$$

Let k be greater than $\dfrac{G - s_{n_1}}{\epsilon}$. Then, $s_{n_k+p_k} > G$, which is contrary to the hypothesis. Hence, by Theorem 16, $s_n \longrightarrow s$. That $s \leqq G$ follows from Theorem 8, corollary 1.

Theorem 18. HYPOTHESES: (i) *s_n is real*; (ii) *$s_n > G$, a fixed number*; (iii) *$s_{n+1} \leqq s_n$ when $n > M$.*

CONCLUSION: *There exists a real number, s, such that $s_n \longrightarrow s$ and $s \geqq G$.*

PROOF: Let $\sigma_n = -s_n$ and $\Gamma = -G$, and apply Theorem 17.

Definition 8. *s_n is said to become infinite if, given any real number N, there exists a number $M > 0$, such that $|s_n| > N$ when $n > M$. We denote this by $s_n \longrightarrow \infty$.*

In case s_n is real, we distinguish between the cases where s_n becomes and remains positive and where it becomes and remains negative by saying, s_n becomes positively infinite ($s_n \longrightarrow +\infty$) and s_n becomes negatively infinite ($s_n \longrightarrow -\infty$).

Theorem 19. HYPOTHESES: (i) *s_n real*; (ii) $s_{n+1} \geqq s_n$; (iii) *s_n does not approach a limit.* CONCLUSION: $s_n \longrightarrow +\infty$.

PROOF: There exists no real number M whatever, such that $s_n \leqq M$ always; otherwise, by Theorem 17, s_n would approach a limit. Moreover, if M is any real number and if $s_n > M$ for $n = m$; then, $s_n > M$ for all values of $n > m$. In other words $s_n \longrightarrow +\infty$.

Theorem 20. HYPOTHESES: (i) *s_n real*; (ii) $s_{n+1} \leqq s_n$; (iii) *s_n does not approach a limit.* CONCLUSION: $s_n \longrightarrow -\infty$.

Proof is omitted.

EXERCISES

15. Prove the theorem:

HYPOTHESES: (i) $s_n^{(1)} \longrightarrow s$, $s_n^{(2)} \longrightarrow s$; (ii) when $n \geqq M$ (fixed), $s_n^{(1)} \leqq \sigma_n \leqq s_n^{(2)}$. CONCLUSION: $\sigma_n \longrightarrow s$.

16. Prove the theorem:

HYPOTHESES: (i) s_n is real and $s_n \longrightarrow s$; (ii) $b > s > c$. CONCLUSION: There exists a fixed M such that when $n > M$, $b > s_n > c$.

17. What, if any, additional hypothesis is necessary to make the following theorem correct?

HYPOTHESES: (i) $s_n \longrightarrow s$; (ii) there exists a sequence, σ_n, such that for any value of n whatever $\sigma_n = s_{n'}$. CONCLUSION: $\sigma_n \longrightarrow$ a limit.

18. Prove the theorem:

HYPOTHESES: (i) $s_n \longrightarrow s$; (ii) there exists a second sequence, σ_n, such that for any n there exists one and only one n', such that $s_n = \sigma_{n'}$ and conversely. CONCLUSION: $\sigma_n \longrightarrow s$.

19. If from an infinite sequence certain values are omitted, the resulting sequence is called a sub-sequence of the first.

Prove the theorem:

HYPOTHESES: (i) $s_n \longrightarrow s$; (ii) σ_n is a sub-sequence of s_n. CONCLUSION: $\sigma_n \longrightarrow s$.

20. Prove the theorem:

HYPOTHESES: $s_n^{(1)} \longrightarrow s^{(1)}$, $s_n^{(2)} \longrightarrow s^{(2)}$, and $|s^{(1)}| > |s^{(2)}|$. CONCLUSION: There exists a fixed number M, such that when $n > M$, $|s_n^{(1)}| > |s_n^{(2)}|$.

21. Prove the theorem:

HYPOTHESES: $s_n \longrightarrow \infty$, σ_n remains finite. CONCLUSION:

$$\frac{\sigma_n}{s_n} \longrightarrow 0.$$

22. Illustrate by figures at least five theorems in this chapter.

23. Prove the theorem:

HYPOTHESES: (i) $s_n^{(1)} \to s^{(1)}, s_n^{(2)} \to s^{(2)}, \ldots, s_n^{(k)} \to s^{(k)}$;
(ii) $F(s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(k)})$
is a rational function with no denominator which vanishes or approaches zero. CONCLUSION:

$$F(s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(k)}) \to F(s^{(1)}, s^{(2)}, \ldots, s^{(k)}).$$

24. If s and q are positive rational numbers, working in the domain of positive real numbers, give a definition for s^q, using the Dedekind section.

25. Using the definition of the last exercise, prove the following theorem:

HYPOTHESES: (i) $s_n \geqq 0$ and rational; (ii) $s_n \to s$; (iii) $q > 0$ and rational. CONCLUSION: $\sqrt[q]{s_n}$ approaches a limit where the radical indicates that root that is positive or zero.

26. Use Exercise 25 to define $\sqrt[q]{s}$, where s is any real number.

27. If $s_n > 0$ and $s_n \to 0$, discuss the behaviour of s_n^{μ} for all possible values of μ.

28. Make up and completely discuss an exercise similar to the last.

Definition. If $s_{n+1} \leqq s_n$ for all values of n or $s_{n+1} \geqq s_n$ for all values of n, s_n is said to be monotonic.

29. Prove the theorem:

If s_n is monotonic, $\dfrac{s_1 + s_2 + \ldots + s_n}{n}$ is monotonic.

30. Compare $\dfrac{s_n}{\sigma_n}$ and $\dfrac{s_1 + s_2 + \ldots + s_n}{\sigma_1 + \sigma_2 + \ldots + \sigma_n}$ as to monotonic character.

31. Give examples of monotonic sequences.

32. Is $n\,(a^{\frac{1}{n}} - 1)$ monotonic? Discuss for all values of a.

33. Prove the theorem:

If $s_{n+1} = \frac{1}{2}(s_n + \sigma_n)$ and $\sigma_{n+1} = \sqrt{s_{n+1}\,\sigma_n}$, where $s_1 > \sigma_1 > 0$; the sequences s_n and σ_n are monotonic and approach a common limit.

34. If $s_{n+1} = \frac{1}{2}(s_n + \sigma_n)$ and $s_{n+1}\,\sigma_{n+1} = s_n\,\sigma_n$, where $s_1 > \sigma_1 > 0$, prove that the sequences s_n and σ_n are monotonic and approach a common limit.

35. If $s_{n+1} = \frac{1}{2}(s_n + \sigma_n)$ and $\sigma_{n+1} = \sqrt{s_n\,\sigma_n}$, where $s_1 > \sigma_1 > 0$, prove that s_n and σ_n are monotonic and approach a common limit.

Definitions. If we are given two sequences s_n and σ_n and if $\dfrac{s_n}{\sigma_n} \to a \neq 0$, we say that s_n is asymptotically proportional to σ_n and sometimes denote it by $s_n \sim \sigma_n$. In case $a = 1$ we say that s_n and σ_n are asymptotically equal and write $s_n \simeq \sigma_n$. In case $\dfrac{s_n}{\sigma_n}$ remains finite, whether it approaches a limit or not, we sometimes write $s_n = O(\sigma_n)$, which is read, s_n is of the order of σ_n. If $\dfrac{s_n}{\sigma_n} \to 0$ a corresponding notation is $s_n = o(\sigma_n)$, which should be read, s_n is of higher order than σ_n.

36, 37, 38. Show that $\sqrt{n^2+1} \simeq n$, $\log(5\,n^2 + 23) \sim \log n$, $\sqrt{n+1} - \sqrt{n} \sim \dfrac{1}{n}$.

39. Give further examples of sequences that are asymptotically equal, that are asymptotically proportional but not asymptotically equal. Give examples of sequences that are of the same order but not asymptotically proportional, of sequences not of the same order. Prove your results in all cases.

Definition. If $s_n \to 0$, s_n is sometimes called a null-sequence. For example, $\dfrac{1}{n}$ is a null-sequence.

40. Under what circumstances is $\dfrac{\log^{\alpha} n}{n^{\beta}}$ a null-sequence?

41. Give other examples of null-sequences, accompanied by proofs.

CHAPTER III

CONVERGENT AND DIVERGENT SERIES

§1. Definitions.

From an infinite series, $a_0, a_1, a_2, \ldots$, an infinite sequence can be formed as follows:

$$s_1 = a_0,\ s_2 = a_0 + a_1,\ s_3 = a_0 + a_1 + a_2, \ldots,$$
$$s_n = a_0 + a_1 + a_2 + \ldots + a_{n-1}, \ldots.$$

Definition 9. *The infinite series, $a_0, a_1, a_2, \ldots$, is said to converge if s_n approaches a limit when n becomes infinite. A series that does not converge is said to diverge.*

Definition 9′. *If s_n approaches a limit, this limit is called the sum of the series. s_n is called the partial sum to n terms, and if the series converges and has the sum s, the difference $s - s_n = r_n$ is called the remainder after n terms.*

A number independent of n will frequently simply be called a constant.

Due to the importance of the number s_n, it is usual to write the series $a_0 + a_1 + a_2 + a_3 + \ldots$, and to say:

$$s = a_0 + a_1 + a_2 + a_3 + \ldots.$$

This will be done here in subservience to established convention. But, it is to be noticed that many things may be of interest about a series besides its sum, and that when we speak of a series we do not thereby refer to its sum. The series may not even converge. It, moreover, is to be noted, that in giving a series it is immaterial which is given, the a_n's or the s_n's; since, from the definition of s_n, $a_0 = s_1$ and $a_n = s_{n+1} - s_n$ when $n > 0$. A convenient notation and one frequently employed is, $s_n = \sum_{n=0}^{n-1} a_n$, and for the infinite series, $\sum_{n=0}^{\infty} a_n$. This last is equivalent to $a_0 + a_1 + a_2 + \ldots$, and does not of itself imply the existence of a sum.

It is also well to explain a notation that is used and which

occasionally will be used in the subsequent pages. It may happen that the numbers of a sequence, a_n, are defined for all positive and negative values of n. The notation,

$$\ldots + a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3 + \ldots,$$

or the equivalent, $\sum_{n=-\infty}^{\infty} a_n$, is frequently used.

Definition 10. $\sum_{n=-\infty}^{\infty} a_n$ *is called a two-way series.*

Definition 11. $\sum_{n=-\infty}^{\infty} a_n$ *is said to converge when and only when $\sum_{n=}^{\infty} a_n$ and $\sum_{n=-1}^{-\infty} a_n$ both converge and its sum is the sum of the sums of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=-1}^{-\infty} a_n$.*

In this book the term 'series' alone does not refer to a two-way series; and in no instance will a two-way series be meant unless it is expressly stated.

§2. Some general theorems on series.

Theorem 21. *A necessary and sufficient condition that $\sum_{n=0}^{\infty} a_n$ converge is that given any $\epsilon > 0$, it is possible to find an M, such that when $n \geqq M$ and $p \geqq 0$,*

$$|a_n + a_{n+1} + \ldots + a_{n+p}| < \epsilon.$$

This theorem is simply a restatement of Theorems 15 and 16 combined with the definition of a convergent series. By introducing the words necessary and sufficient, both theorems are combined into one.

Corollary. *When $\sum_{n=0}^{\infty} a_n$ converges, $a_n \to 0$.*

The converse of this corollary is not true, as many examples will testify.

Theorem 22. HYPOTHESIS: $\sum_{n=0}^{\infty} a_n$ *converges to s.* CON-

CLUSION: $\sum_{n=k}^{\infty} a_n$, *where k is any fixed integer, converges to $s - s_k$.*

PROOF. When $n > k$ we have

$$s_n = a_0 + \ldots + a_{k-1} + a_k + \ldots + a_{n-1}.$$

Let $n' = n - k$ and $\sigma_{n'} = a_k + \ldots + a_{n-1}$. Then $\sigma_{n'} = s_n - s_k$. s_k is constant and $s_n \to s$. Hence, by Theorems 4 and 12, $\sigma_{n'} \to s - s_k$ which is what we desired to prove.

The converse of this theorem is readily proved, but due to its simplicity is omitted.

Theorem 23. HYPOTHESIS: $\sum_{n=0}^{\infty} a_n$ *diverges.* CONCLUSION: $\sum_{n=k}^{\infty} a_n$ *diverges, k being any fixed integer.*

PROOF: Suppose the contrary and use the same notation under Theorem 22. Then $s_n = \sigma_{n'} + s_k$. Since s_k is a constant and $\sigma_{n'}$ approaches a limit, s_n approaches a limit, which is contrary to the hypothesis, establishing the theorem.

The converse of this theorem is also readily proved.

Theorems 22 and 23 and their converses can be in part summarized by saying: The convergence or divergence of series is not affected by the removal or addition of a fixed number of terms.

As a corollary to this it follows that any finite number of terms in a series can have their values changed without affecting convergence or divergence.

Theorem 24. HYPOTHESIS: $\sum_{n=0}^{\infty} a_n = s$. CONCLUSION: $\sum_{n=0}^{\infty} ca_n = cs$, *where c is any constant.*

PROOF: Let $\quad s_n = a_0 + a_1 + \ldots + a_{n-1}$

and $\quad t_n = ca_0 + ca_1 + \ldots + ca_{n-1}$.

Then $t_n = cs_n$. Consequently, by Theorem 5, $t_n \to cs$.

Theorem 25. HYPOTHESIS: $\sum_{n=0}^{\infty} a_n$ *diverges.* CONCLUSION: $\sum_{n=0}^{\infty} ca_n$, *where c is a constant different from zero, diverges.*

PROOF: Let s_n and t_n be as above. $s_n = \dfrac{t_n}{c}$. Suppose t_n approaches a limit, then s_n will approach a limit also by Theorem 5. This is contrary to the hypothesis.

Theorem 26. HYPOTHESIS: $\sum_{n=0}^{\infty} |a_n|$ *converges.* CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges and* $\left|\sum_{n=0}^{\infty} a_n\right| \leqq \sum_{n=0}^{\infty} |a_n|$.

PROOF: Given any $\epsilon > 0$, it is possible to choose an $M > 0$ such that $|a_n| + |a_{n+1}| + \ldots + |a_{n+p}| < \epsilon$ when $n \geqq M$ and $p \geqq 0$. See Theorem 21. But,

$$|a_n + a_{n+1} + \ldots + a_{n+p}| \leqq |a_n| + |a_{n+1}| + \ldots + |a_{n+p}|.$$

Hence, when $n \geqq M$ and $p \geqq 0$, $|a_n + a_{n+1} + \ldots + a_{n+p}| < \epsilon$ and by Theorem 21 $\sum_{n=0}^{\infty} a_n$ converges.

To prove the second part of the theorem, let

$s_n = a_0 + a_1 + \ldots + a_{n-1}$ and $t_n = |a_0| + |a_1| + \ldots + |a_{n-1}|$.

Then $|s_n| \leqq t_n$. Hence, $\lim |s_n| \leqq \lim t_n$, that is

$$\left|\sum_{n=0}^{\infty} a_n\right| \leqq \sum_{n=0}^{\infty} |a_n|.$$

Theorem 27. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ *converges.* CONCLUSION: $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ *converges also.*

PROOF: Suppose $|s_n| = |a_1 + \ldots + a_n| < G$ always.

$$\left|\sum_{n=\mu}^{\nu} \frac{a_n}{n}\right| = \left|(s_\mu - s_{\mu-1})\frac{1}{\mu} + \ldots + (s_\nu - s_{\nu-1})\frac{1}{\nu}\right|$$

$$= \left| -s_{\mu-1} \cdot \frac{1}{\mu} + s_\mu\left(\frac{1}{\mu} - \frac{1}{\mu+1}\right) + \ldots + s_{\nu-1}\left(\frac{1}{\nu-1} - \frac{1}{\nu}\right) + s_\nu \frac{1}{\nu}\right| \leqq G \cdot \frac{2}{\mu} < \epsilon, \text{ if } \mu > \frac{2G}{\epsilon}.$$

Hence, given any $\epsilon > 0$, it is possible to choose an $M \geqq \frac{2G}{\epsilon}$ such that when $\nu \geqq \mu > M, |s_\nu - s_\mu| < \epsilon$, which, by Theorem 21, establishes convergence.

The following theorem is not directly on infinite series, but is introduced as a lemma. Its applications in the study of series are immediate and numerous.

Theorem 28. HYPOTHESES: (i) $\epsilon_0 \geqq \epsilon_1 \geqq \epsilon_2 \geqq \dots \geqq \epsilon_p \geqq 0$; (ii) $s_0 = a_0, s_1 = a_0 + a_1, \dots, s_p = a_0 + a_1 + \dots + a_p$; (iii) *$A$ is as large as any of the numbers* $|s_0|, |s_1|, \dots, |s_p|$; (iv) $S = \epsilon_0 a_0 + \epsilon_1 a_1 + \dots + \epsilon_p a_p$. CONCLUSION: $|S| \leqq A\epsilon_0$.

PROOF:

(1) $S = s_0(\epsilon_0 - \epsilon_1) + s_1(\epsilon_1 - \epsilon_2) + \dots + s_{p-1}(\epsilon_{p-1} - \epsilon_p) + s_p \epsilon_p$.

Hence
$$|S| \leqq |s_0|(\epsilon_0 - \epsilon_1) + |s_1|(\epsilon_1 - \epsilon_2) + \dots + |s_{p-1}|(\epsilon_{p-1} - \epsilon_p) + |s_p|\epsilon_p \leqq A[\epsilon_0 - \epsilon_1 + \epsilon_1 - \epsilon_2 + \dots + \epsilon_{p-1} - \epsilon_p + \epsilon_p] = \epsilon_0 A.$$

Corollary. *In case $a_0, a_1, \dots, a_p$ are real and B denotes the smallest of the numbers, $s_0, s_1, \dots, s_p$, and A the largest, we readily conclude from equation* (1) *the more restrictive inequality,*
$$\epsilon_0 B \leqq S \leqq \epsilon_0 A,$$
which is the form in which this theorem is usually given.

Theorem 29. *Consider* $\sum_{n=0}^{\infty} a_n$ *where* $s_n = a_0 + \dots + a_n$. HYPOTHESES: (i) $|s_n| \leqq G$ *a number independent of* n; (ii) $\epsilon_0 \geqq \epsilon_1 \geqq \epsilon_2 \dots$; (iii) $\epsilon_n \longrightarrow 0$. CONCLUSION: $\sum_{n=0}^{\infty} \epsilon_n a_n$ *converges.*

PROOF: $|a_{n+1} + \dots + a_{n+p}| = |s_{n+p} - s_n| \leqq 2G$. Hence, by the previous theorem, $|\epsilon_{n+1} a_{n+1} + \dots + \epsilon_{n+p} a_{n+p}| \leqq 2G\epsilon_{n+1}$, which approaches zero as $n \longrightarrow \infty$, establishing the theorem. See Theorem 21.

The theorems in this chapter, with the possible exception of 27 and 29, are fundamental in the developments that are to follow. It is to be noticed that no hypothesis whatever is made as to the nature of the terms of the series. They may

be complex or real. Other theorems of this general character might be given and some will be subsequently. It is thought best, however, to proceed as explained at the beginning of the next chapter.

EXERCISES

Definition. A series of the form
$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots,$$
is called a telescopic series.

42. Prove: Any series can be written as a telescopic series.

43. Prove: A telescopic series of the form given above converges when and only when $\lim_{n \to \infty} a_n$ exists.

44. Write at least five of the series of exercises 53–100 as telescopic series.

45. Prove the theorem: HYPOTHESIS: $\rho_n > \rho_{n+1} \longrightarrow \lambda$ and $|s_n| < M$. CONCLUSION: $\sum_{n=0}^{\infty} (\rho_n - \rho_{n+1}) s_n$ converges and
$$\left| \sum_{n=0}^{\infty} (\rho_n - \rho_{n+1}) s_n \right| \leqq M(\rho_0 - \lambda).$$

46. Is the following theorem true or false? $\sum_{n=1}^{\infty} |a_n|$ converges if every series formed from the terms of $\sum_{n=1}^{\infty} a_n$, no term used more than once, converges.

47. By theorems in this chapter, examine $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ for convergence.

48. Prove $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent by means of Theorem 21. Prove $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent by the same theorem.

49. Prove the following theorem. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n$ converges; (ii) ϵ_n is a monotonic sequence; (iii) $| \epsilon_n | < \blacksquare$ (a constant). CONCLUSION: $\sum_{n=1}^{\infty} \epsilon_n a_n$ converges.

50. Examine $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{2}{3^2} + \frac{1}{4} - \frac{3}{4^2} + \dots$, for convergence by comparison with $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$, according to the theorem of Exercise 49. Apply Exercise 49 to other series.

51. If $na_n \to 0$, what can you say as to convergence of $\sum_{n=1}^{\infty} a_n$? If $\sum_{n=1}^{\infty} a_n$ converges, is it necessary that $na_n \to 0$?

52. Is the following theorem true or false? HYPOTHESES: $na_n \to 0$ and $\sum_{n=1}^{\infty} n\,(a_n - a_{n+1})$ converges. CONCLUSION: $\sum_{n=1}^{\infty} a_n$ converges.

CHAPTER IV

SERIES WHOSE TERMS ARE POSITIVE

In the present chapter all the terms of every series which enters are positive real numbers. This is stated here once and for all and a separate statement of the fact is not made for each theorem. Methods of testing for convergence or divergence are developed.

It need hardly be remarked, that if in an investigation for convergence or divergence all the terms of a series are negative, the signs of all terms can be changed and the resulting series examined. See Theorem 24.

Theorem 30. HYPOTHESIS: $\sum_{n=0}^{n-1} a_n < G$, *a fixed number.*

CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges.*

PROOF: This theorem is just a restatement of Theorem 17; and for a proof see that theorem.

A carefully constructed figure will make the theorem appear extremely likely without the formality of proof. The reader is advised to make such a figure here and frequently with reference to other theorems.

Figures aid in foreseeing conclusions and lead to a quick understanding of the steps of a proof.

Theorem 31. HYPOTHESES:

$$\text{(i)} \sum_{n=0}^{\infty} a_n \textit{ converges}; \quad \text{(ii)} \ a_n \geqq a_{n+1}.$$

CONCLUSION: $na_n \to 0$.

PROOF: Let n be even.

$$\sum_{\nu=\frac{n}{2}}^{n} a_\nu \geqq \frac{n+2}{2} a_n > \frac{n}{2} a_n,$$

but by Theorem 21 $\sum_{\nu=\frac{n}{2}}^{n} a_\nu \to 0$, and hence $\frac{n}{2} a_n \to 0$, and hence $na_n \to 0$.

Let n be odd.

$$\sum_{\nu=\frac{n-1}{2}}^{n} a_\nu \geqq \frac{n+3}{2} a_n > \frac{n}{2} a_n,$$

and hence, as above, $na_n \to 0$.

Theorem 32. HYPOTHESIS: *$\sum_{n=0}^{\infty} a_n$ diverges.* CONCLUSION: *$s_n \to +\infty$.*

It is not unusual to see the conclusion of this theorem expressed by the statement: $\sum_{n=1}^{\infty} a_n$ diverges to plus infinity.

PROOF: s_n never decreases. Suppose that there exists a number G such that $s_n < G$ always. By Theorem 30

$$\sum_{n=1}^{\infty} a_n$$

converges. This is contrary to the hypothesis.

Theorem 33. HYPOTHESIS: *$\sum_{n=0}^{\infty} a_n$ converges.* CONCLUSION: *$\sum_{n=0}^{\infty} a_{\lambda_n}$ converges, where the λ_n's are any set of integers chosen from the integers n.*

PROOF: Let $\sum_{n=0}^{\infty} a_n = s$, $s_n = a_0 + \ldots + a_{n-1}$, and

$$\sigma_n = a_{\lambda_1} + \ldots + a_{\lambda_{n-1}}.$$

Now $s_n < s$, and given any σ_m it is possible to choose n so large that $s_n > \sigma_m$. Consequently $\sigma_m < s$ always, and consequently $\sum_{n=1}^{\infty} a_{\lambda_n}$ converges.

Theorem 34. HYPOTHESES: (i) *$\phi(x) \geqq 0$ when $x \geqq a$;* (ii) *$\phi(x+\delta) \leqq \phi(x)$ when $x \geqq a$ and $\delta > 0$;*

$$\text{(iii)} \int_a^b \phi(x)\,dx \to G \text{ when } b \to \infty.$$

CONCLUSION: *$\sum_{n=a}^{\infty} \phi(n)$ converges.*

PROOF:

$$s_n = \phi(a) + \phi(a+1) + \ldots + \phi(a+n) \leqq \phi(a) + \int_a^{a+n} \phi(x)\,dx.$$

This follows from the definition * of an integral.

$$\int_a^{a+n} \phi(x)\,dx$$

does not decrease as n increases, and approaches G when $n \to \infty$. Hence $s_n \leqq \phi(a) + G$. Hence, by Theorem 30 the series converges.

Theorem 35. HYPOTHESES: *Same as under 34 except* (iii) *where $\int_a^b \phi(x)\,dx \to \infty$ when $b \to \infty$.* CONCLUSION: *$\sum_{n=a}^{\infty} \phi(n)$ diverges.*

PROOF: $s_{n-1} \geqq \int_a^{a+n} \phi(x)\,dx$ and consequently $s_n \to \infty$ when $n \to \infty$.

Theorem 36. HYPOTHESES: *Given two series, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, where* (i) *$\sum_{n=0}^{\infty} a_n$ converges and* (ii) *when $n \geqq k$, $b_n \leqq a_n$.* CONCLUSION: *$\sum_{n=0}^{\infty} b_n$ converges.*

PROOF: We may discard from consideration all terms before the k-th by Theorems 22 and 23 and their converses.

Let $s_n = a_k + a_{k+1} + \ldots + a_{k+n}$

and $t_n = b_k + b_{k+1} + \ldots + b_{k+n}$.

* Riemann definition alone considered here and in other places unless the contrary is expressly stated.

Choose G so that $G > s_n$ always. Then $t_n \leqq s_n < G$. But t_n never decreases, and consequently approaches a limit by Theorem 17, that is, $\sum_{n=k}^{\infty} b_n$ converges and consequently $\sum_{n=0}^{\infty} b_n$ converges.

Theorem 37. HYPOTHESES: *Given two series,* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$, *where* (i) $\sum_{n=0}^{\infty} a_n$ *diverges and* (ii) *when* $n \geqq k$, $b_n \geqq a_n$. CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *diverges.*

Proof is omitted due to the similarity of this theorem to the preceding.

An immediate result of the preceding theorems is the following:

Theorem 38. I. HYPOTHESES: $\sum_{n=0}^{\infty} a_n$ *converges and* $\frac{a_n}{b_n} > \epsilon > 0$ *when* $n \geqq k$, *a fixed number.* CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *converges.*

II. HYPOTHESES: $\sum_{n=0}^{\infty} a_n$ *diverges and* $\frac{b_n}{a_n} > \epsilon > 0$ *when* $n \geqq k$. CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *diverges.*

PROOF: I. $a_n > \epsilon b_n$, $n \geqq k$. Hence $\sum_{n=0}^{\infty} \epsilon b_n$ converges, and as ϵ is independent of n, $\sum_{n=0}^{\infty} b_n$ converges.

II. Similarly $b_n > a_n \epsilon$, and as $\sum_{n=0}^{\infty} a_n \epsilon$ diverges so does $\sum_{n=0}^{\infty} b_n$.

Corollary to I. HYPOTHESIS: $\frac{a_n}{b_n} \rightarrow \eta > 0$. CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *converges.*

Corollary to II. HYPOTHESIS: $\frac{b_n}{a_n} \rightarrow \eta > 0$. CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *diverges.*

What amounts to nothing more than another wording of the preceding theorems is the following.

Theorem 39. I. HYPOTHESES: (i) $\sum_{n=0}^{\infty} a_n$ *converges;* (ii) $b_n < G$ *a fixed number.* CONCLUSION: $\sum_{n=0}^{\infty} a_n b_n$ *converges.*

II. HYPOTHESES: (i) $\sum_{n=0}^{\infty} a_n$ *diverges;* (ii) $b_n > \epsilon > 0$ *where* ϵ *is fixed.* CONCLUSION: $\sum_{n=0}^{\infty} a_n b_n$ *diverges.*

We shall now give a few series that can be used as comparison series (see Theorems 36, 37, 38, and 39). It is clear, however, that the more series whose convergence or divergence has been determined, the more comparison series we have, although not necessarily all of the same usefulness.

The geometric series

(1) $$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots, \quad a \neq 0,$$

converges if $r < 1$ and diverges if $r \geqq 1$.

PROOF: When $r \neq 1$, $s_n = \frac{a - ar^n}{1 - r}$, from which the conclusions of the theorem are immediate.

When $r = 1$ the series reduces to $a + a + a + \ldots$ which diverges, since $a \neq 0$.

(2) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$$

converges.

PROOF: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Hence $s_n = 1 - \frac{1}{n+1} \to 1$.

(3) $$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

converges when $p > 1$ and diverges when $p \leqq 1$.

PROOF: Apply the integral test of Theorems 34 and 35 and the stated results are readily established. It is a useful exercise to examine this series by other methods.

Let $\log_2 n = \log\log n, \ldots,$

$\log_m n = \log \log_{m-1} n,$

and let $A_m(n) = \log n . \log_2 n \ldots . \log_m n.$

Also let $e_2 = e^e,\ e_3 = e_2^e,\ \ldots,\ e_m = e_{m-1}^e.$

Then let $a > e_{m+1}$.

(4) $\sum_{n=a}^{\infty} \frac{1}{n A_m(n) \log_{m+1}^{\rho} n}$ diverges if $\rho \leqq 1$ and converges if $\rho > 1$.

PROOF: Apply the integral test of Theorems 34 and 35.

$$\int_a^b \frac{dx}{x A_m(x) \log_{m+1}^{\rho} x} = \frac{1}{1-\rho}[\log_{m+1}^{1-\rho} b - \log_{m+1}^{1-\rho} a] \text{ if } \rho \neq 1.$$

When $\rho < 1$ this $\to \infty$ when $b \to \infty$, and if $\rho > 1$ approaches a limit when $b \to \infty$. If $\rho = 1$, replace $m+1$ by m' and we have the series $\sum_{n=a}^{\infty} \frac{1}{n A_{m'}(n)}$, which diverges, for it is of the same form as (4) with $\rho = 0$.

Theorem 40. *Consider* $\sum_{n=0}^{\infty} a_n$. I. HYPOTHESIS:

$$0 \leqq \sqrt[n]{a_n} < r < 1$$

when $n > k$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $\sqrt[n]{a_n} \geqq 1$ *when* $n > k$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *diverges.*

PROOF: I. Here $a_n < r^n$, and by comparison with the geometric series $\sum_{n=0}^{\infty} a_n$ converges.

II. Here $a_n \geqq 1$, and by Theorem 21, corollary, the series diverges.

Corollary. I. HYPOTHESIS: $\overline{a_n} \to l < 1$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $\sqrt[n]{a_n} \to l > 1$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *diverges.*

The corollary makes no statement as to the case where $l = 1$ and the approach not from above. Examples can be given where the series converges, also examples where it diverges. For instance, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p = 1$ as we have already seen, but in both instances $\sqrt[n]{\frac{1}{n^p}} \to 1$ and $\sqrt[n]{\frac{1}{n^p}} < 1$.

Theorem 41. *As usual, consider* $\sum_{n=0}^{\infty} a_n$. I. HYPOTHESIS: $\frac{a_{n+1}}{a_n} < r < 1$ *when* $n \geqq k$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $\frac{a_{n+1}}{a_n} \geqq 1$ *when* $n \geqq k$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *diverges.*

PROOF. I. $\frac{a_{k+1}}{a_k} < r$. Hence $a_{k+1} < ra_k$.

$\frac{a_{k+2}}{a_{k+1}} < r$. Hence $a_{k+2} < ra_{k+1} < r^2 a_k$.

. . . .

$\frac{a_{k+j+1}}{a_{k+j}} < r$. Hence $a_{k+j+1} < ra_{k+j} < \ldots < r^{j+1} a_k$.

But $a_k r + a_k r^2 + \ldots$ converges as it is a geometric series with $r < 1$. Hence by Theorem 36, $\sum\limits_{n=0} a_n$ converges.

II. Here $a_{k+1} \geqq a_k, \ldots, a_{k+j+1} \geqq a_k$. Since $a_k > 0$, the series diverges by Theorem 21, corollary.

Corollary. *Suppose* $\dfrac{a_{n+1}}{a_n} \to l$. I. HYPOTHESIS: $l < 1$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $l > 1$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *diverges.*

In case $l = 1$ the series may either converge or diverge, witness $\sum\limits_{n=0}^{\infty} \dfrac{1}{n^p}$ when $p > 1$ and when $p = 1$.

Theorem 42. I. HYPOTHESIS: $n\left(\dfrac{a_n}{a_{n+1}} - 1\right) > r > 1$ *when* $n \geqq k$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $n\left(\dfrac{a_n}{a_{n+1}} - 1\right) \leqq 1$ *when* $n \geqq k$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *diverges.*

PROOF: I. Let $r = 1 + \epsilon$, then when $n \geqq k$,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 + \epsilon.$$

That is $$na_n - (n+1)\,a_{n+1} > \epsilon a_{n+1} > 0.$$

In other words, na_n, which is greater than zero, decreases monotonically. Hence na_n approaches a limit by Theorem 18. Denote this limit by α.

$$\sum_{n=0}^{m-1} (na_n - (n+1)\,a_{n+1}) = -ma_m \to -\alpha.$$

That is, $\sum\limits_{n=0}^{\infty} (na_n - (n+1)\,a_{n+1})$ converges. Hence

$$\sum_{n=0}^{\infty} \epsilon a_{n+1} \text{ and hence } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

See Theorems 24 and 36.

II. $\qquad na_n - (n+1)\,a_{n+1} \leqq 0, \quad n \geqq k.$

Hence $\qquad ka_k \leqq na_n, \quad n > k.$

Hence $a_n \geqq a_k \cdot \dfrac{k}{n}$. But $a_k \cdot \dfrac{k}{n}$ is the general term of a divergent series. Hence, by Theorem 37, $\sum\limits_{n=0}^{\infty} a_n$ diverges.

Corollary 1. *Suppose* $n\left(\dfrac{a_n}{a_{n+1}} - 1\right) \to l$. I. HYPOTHESIS: $l > 1$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: $l < 1$. CONCLUSION: $\sum\limits_{n=0}^{\infty} a_n$ *diverges.*

Corollary 2. *If* $\dfrac{a_n}{a_{n+1}} = 1 + \dfrac{\alpha}{n} + \dfrac{A_n}{n^2}$, *where* α *is a constant and* $|A_n| < K$, *then* $\sum\limits_{n=0}^{\infty} a_n$ *converges if* $\alpha > 1$ *and diverges if* $\alpha < 1$.

The theorem which has just been proved will be illuminated if we apply it to the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$. Here

$$n\left[\left(\frac{n+1}{n}\right)^p - 1\right] \to p,$$

and we draw the conclusions which we already know relative to convergence or divergence of the series. Theorem 41, the test ratio test, does not give the results. Moreover Theorem 41 was established as a consequence of known facts relative to the geometric series. A parallel situation exists with reference to Theorem 42. We have given an independent

proof and investigated $\sum_{n=1}^{\infty} \frac{1}{n^p}$ by means of it. This process can be reversed and the test made a consequence of our knowledge of $\sum_{n=1}^{\infty} \frac{1}{n^p}$. However, the whole situation will be made clearer after the following two fundamental theorems have been established.

Theorem 43. HYPOTHESES:

(i) $\sum_{n=0}^{\infty} a_n$ *converges*; (ii) $\sum_{n=0}^{\infty} b_n$ *is a second series*,

and $\frac{b_{n+1}}{b_n} \leqq \frac{a_{n+1}}{a_n}$ *when* $n \geqq k$. CONCLUSION:

$$\sum_{n=0}^{\infty} b_n \textit{ converges.}$$

PROOF: $\frac{b_{k+1}}{b_k} \leqq \frac{a_{k+1}}{a_k}$, $\frac{b_{k+2}}{b_{k+1}} \leqq \frac{a_{k+2}}{a_{k+1}}$, $\ldots$, $\frac{b_{k+m+1}}{b_{k+m}} \leqq \frac{a_{k+m+1}}{a_{k+m}}$.

Multiply these inequalities together and we have

$$\frac{b_{k+m+1}}{b_k} \leqq \frac{a_{k+m+1}}{a_k} \quad \text{or} \quad b_{k+m+1} \leqq \frac{b_k}{a_k} a_{k+m+1}.$$

But the series $\sum_{m=0}^{\infty} \frac{b_k}{a_k} a_{k+m+1}$ converges. Hence, by Theorem 36, $\sum_{n=0}^{\infty} b_n$ converges also.

What might be given as a second half of the theorem that has just been proved is the following theorem. Its proof is so similar to what we have just given that it is omitted.

Theorem 44. HYPOTHESES:

(i) $\sum_{n=0}^{\infty} a_n$ *diverges*; (ii) $\frac{b_{n+1}}{b_n} \geqq \frac{a_{n+1}}{a_n}$ *when* $n \geqq k$.

CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *diverges.*

To apply these theorems to the proof of Theorem 42, let

$$n\left[\frac{a_n}{a_{n+1}} - 1\right] > r > 1$$

when $n \geqq k$, and let $1 < p < r$. We have remarked that

$$n\left[\left(1+\frac{1}{n}\right)^p - 1\right] \to p.$$

Hence, when $n \geqq M \geqq k$,

$$n\left[\frac{a_n}{a_{n+1}} - 1\right] > n\left[\left(\frac{n+1}{n}\right)^p - 1\right]$$

$$\text{or} \quad \frac{a_{n+1}}{a_n} < \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}$$

and we draw the first conclusion of Theorem 42. The second conclusion is drawn in exactly the same way.

An extension of Theorem 42 is given by the following theorem.

Theorem 45. I. HYPOTHESIS:

$$\left[n \log n \frac{a_n}{a_{n+1}} - (n+1)\log(n+1)\right] > r > 0$$

when $n \geqq k$. CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *converges.*

II. HYPOTHESIS:

$$\left[n \log n \frac{a_n}{a_{n+1}} - (n+1)\log(n+1)\right] \leqq 0$$

when $n \geqq k$. CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *diverges.*

PROOF: We follow exactly the proof given of 42. If the second proof of 42 is given, the ratio $\frac{a_{n+1}}{a_n}$ is compared with the ratio

$$\frac{\frac{1}{(n+1)(\log(n+1))^p}}{\frac{1}{n(\log n)^p}}.$$

Changes are so slight as not to warrant a repetition.

Further extensions are quite possible but are left as an exercise.

Theorem 46. HYPOTHESIS: $g_1, g_2, \ldots$, are a sequence of positive numbers such that

$$g_n \frac{a_n}{a_{n+1}} - g_{n+1} > \epsilon > 0$$

when $n \geqq k$. CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *converges.*

PROOF: $g_n a_n - g_{n+1} a_{n+1} > \epsilon a_{n+1} > 0$ when $n \geqq k$. Hence $g_n a_n$ approaches a limit. Therefore,

$$\sum_{n=0}^{\infty} (g_n a_n - g_{n+1} a_{n+1})$$

converges. Hence $\sum_{n=0}^{\infty} \epsilon a_n$ and hence $\sum_{n=0}^{\infty} a_n$ converges.

The following is the corresponding theorem relative to divergence.

Theorem 47. HYPOTHESIS: $\sum_{n=0}^{\infty} \frac{1}{g_n}$ *diverges, and when* $n > \nu$, $g_\nu a_\nu < g_n a_n$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *diverges.*

PROOF: $a_n > \frac{g_\nu a_\nu}{g_n}$, but $g_\nu a_\nu$ is a constant. Hence

$$\sum_{n=0}^{\infty} \frac{g_\nu a_\nu}{g_n}$$

diverges and hence, by comparison, according to Theorem 37, $\sum_{n=0}^{\infty} a_n$ diverges.

Some of the tests previously given can easily be derived as special cases under the last two theorems. For example: let $g_n = 1$ and we have the test ratio test, Theorem 41; let $g_n = n$ and an equivalent of Theorem 42 results, and let $g_n = n \log n$ and we obtain the first extension of Theorem 42.

Theorem 48. I. HYPOTHESIS: *when* $n < k$,

$$(1 - \sqrt[n]{a_n}) \frac{n}{\log n} > r > 1.$$

CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *converges.*

II. HYPOTHESIS: *When* $n > k$,

$$(1 - \sqrt[n]{a_n}) \frac{n}{\log n} \leqq 1.$$

CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *diverges.*

PROOF: Let $a_n = n^{-p_n}$, thus defining p_n. Then when $a_n < 1$,

$$\sqrt[n]{a_n} = n^{-\frac{p_n}{n}} = e^{-\frac{p_n}{n}\log n} > 1 - \frac{p_n}{n} \log n.*$$

Hence $$p_n > (1 - \sqrt[n]{a_n}) \frac{n}{\log n}.$$

Consequently if $(1 - \sqrt[n]{a_n}) \frac{n}{\log n} > r > 1$,

$p_n > r > 1$. Hence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^{-p_n}$ converges. See Example 3, page 30.

II. If $(1 - \sqrt[n]{a_n}) \frac{n}{\log n} \leqq 1$ when $n > k$,

$$a_n \geqq \left(1 - \frac{1}{n} \log n\right)^n$$

when $n > k$. But $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n} \log n\right)^n$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $n\left(1 - \frac{1}{n} \log n\right)^n \rightarrow 1$. See Theorem 38. Consequently, by Theorem 37, $\sum_{n=1}^{\infty} a_n$ diverges.

* To prove this, apply the 'Law of the Mean' to e^{-x}, $x > 0$.

$$e^{-x} = 1 - x e^{-\xi} \quad \text{where } 0 < \xi < x.$$

But $0 < e^{-\xi} < 1$. Hence $e^{-x} > 1 - x$.

Corollary. If $\lim_{n\to\infty}(1-\sqrt[n]{a_n})\dfrac{n}{\log n}=l$, then $\sum_{n=1}^{\infty}a_n$ converges in case $l>1$, and diverges in case $l<1$.

The foregoing theorems in this chapter give the more common tests for convergence or divergence, and by means of some one or more of them the great majority of series with positive terms can be satisfactorily investigated. They are all well known theorems occurring at numerous places in mathematical literature. A few additional theorems possibly not quite so well known but which are sometimes useful are now given.

Theorem 49. HYPOTHESIS: $\sum_{n=1}^{\infty}a_n$ *diverges.* CONCLUSION:

$$\sum_{n=1}^{\infty}\frac{a_n}{a_1+\dots+a_n}\ \textit{diverges.}$$

PROOF:

$$\sum_{\nu=n+1}^{n+p}\frac{a_\nu}{a_1+\dots+a_\nu}\geqq\frac{a_{n+1}+\dots+a_{n+p}}{a_1+\dots+a_{n+p}}=1-\frac{a_1+\dots+a_n}{a_1+\dots+a_{n+p}}.$$

Now suppose $\sum_{n=1}^{\infty}\dfrac{a_n}{a_1+\dots+a_n}$ convergent. Then

$$\sum_{\nu=n+1}^{n+p}\frac{a_\nu}{a_1+\dots+a_\nu}\to 0$$

when $n\to\infty$. Consequently it is possible to choose an m such that when $n\geqq m$,

$$\left|1-\frac{a_1+\dots+a_n}{a_1+\dots+a_{n+p}}\right|<\tfrac{1}{2}\ \text{for example.}$$

But for any fixed n when $p\to\infty$

$$\left|1-\frac{a_1+\dots+a_n}{a_1+\dots+a_{n+p}}\right|\to 1,\ \text{a contradiction.}$$

An example of a series that can be readily investigated by means of this theorem is the following,

$$\sum_{n=1}^{\infty}\frac{\log(n+1)-\log n}{\log(n+1)}.$$

Theorem 50. HYPOTHESES:

(i) $0<M_1<M_2<\dots<M_n<\dots$; (ii) $M_n\to\infty$.

CONCLUSION: $\sum_{n=1}^{\infty}\dfrac{M_{n+1}-M_n}{M_{n+1}M_n}$ *converges.*

PROOF:

$$\sum_{n=1}^{n}\frac{M_{n+1}-M_n}{M_{n+1}M_n}=\sum_{n=1}^{n}\left(\frac{1}{M_n}-\frac{1}{M_{n+1}}\right)=\frac{1}{M_1}-\frac{1}{M_{n+1}}\to\frac{1}{M_1},$$

which not only establishes convergence but gives the value of the sum of the series.

A special case of this theorem which has already been investigated is that when $M_n=n$, and the series proved convergent is $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$.

Theorem 51. HYPOTHESES:

(i) $0<M_1<M_2<\dots<M_n<\dots$; (ii) $M_n\to\infty$.

CONCLUSION: $\sum_{n=1}^{\infty}\dfrac{M_{n+1}-M_n}{M_{n+1}M_n^\rho}$, $\rho>0$ *converges.*

PROOF: Let $\dfrac{M_n}{M_{n+1}}=q_n$. Then $0<q_n<1$. Assume $\rho<1$.

From the preceding theorem

$$\sum_{n=1}^{\infty}\frac{M_{n+1}^\rho-M_n^\rho}{M_{n+1}^\rho M_n^\rho}$$

converges. But

$$\frac{M_{n+1}^\rho-M_n^\rho}{M_{n+1}^\rho M_n^\rho}=\frac{M_{n+1}-M_n}{M_{n+1}M_n^\rho}\cdot\frac{1-q_n^\rho}{1-q_n}.$$

Moreover $\dfrac{1-q_n^\rho}{1-q_n}>\rho$. Hence

$$\frac{M_{n+1}-M_n}{M_{n+1}M_n^\rho}<\frac{1}{\rho}\,\frac{M_{n+1}^\rho-M_n^\rho}{M_{n+1}^\rho M_n^\rho}.$$

Consequently by Theorem 36 the series to be tested converges.

The case that $\rho=1$ has been treated in the previous theorem. If $\rho>1$ the theorem follows from the previous theorem and Theorem 36.

Corollary. $\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_{n+1}^{1+\rho}}$ *converges.*

Assume* as a preliminary to the next theorem the following inequality. Let y and z be two positive numbers, $y > z$. Moreover, when x is any positive number large enough to give the symbols a meaning, let, as previously,

$$\log_1 x = \log x,\ \log_2 x = \log \log_1 x,\ \ldots,$$

and let $\quad A_m x = (\log_1 x)(\log_2 x) \ldots (\log_m x).$

Then $$\frac{y-z}{y A_{m-1} y} < \log_m y - \log_m z < \frac{y-z}{z A_{m-1} z}.$$

Theorem 52. HYPOTHESES:

$$M_{n+1} > M_n > 0,\ M_n \to \infty,\ \text{and } \rho > 0.$$

CONCLUSIONS:

(i) $$\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_{n+1} M_n^{\rho}},\quad \sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_{n+1}(\log M_{n+1})(\log^{\rho} M_n)},\ \ldots,$$

$$\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_{n+1}(A_m M_{n+1})(\log_m^{\rho} M_n)} \text{ all converge;}$$

(ii) $$\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_n},\quad \sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_n \log M_n},\ \ldots,$$

$$\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_n A_m M_n} \text{ all diverge.}$$

We, of course, assume M_n large enough for the symbols to have a meaning.

PROOF: The first series given in conclusion (i) has already been treated (Theorem 50). From the hypotheses and the fact that the logarithm is an increasing function becoming infinite with its argument

$$\log_m M_{n+1} > \log_m M_n \quad \text{and} \quad \log_m M_n \to \infty$$

when $n \to \infty$. From this $\log_m M_n$ can replace M_n of the theorem, and consequently, as just remarked,

$$\sum_{n=1}^{\infty} \frac{\log_m M_{n+1} - \log_m M_n}{\log_m M_{n+1} \log_m^{\rho} M_n}$$

is convergent.

* See, for example, Jordan: *Cours d'Analyse*, T. 1, p. 291.

Apply the inequality preceding this theorem and we see that the n-th term of this series is greater than

$$\frac{M_{n+1}-M_n}{M_{n+1}(A_{m-1} M_{n+1}) \log_m M_{n+1} \log_m^{\rho} M_n} = \frac{M_{n+1}-M_n}{M_{n+1}(A_m M_{n+1})(\log_m^{\rho} M_n)}.$$

Then by Theorem 36 we complete conclusion (i).

Next $$\sum_{n=1}^{\infty} (\log_m M_{n+1} - \log_m M_n)$$

diverges; for, denoting the sum of the first n terms by s_n, we have $s_n = \log_m M_{n+1} - \log_m M_1$, and hence $s_n \to \infty$. The terms of this series are each less than the corresponding term of

$$\sum_{n=1}^{\infty} \frac{M_{n+1}-M_n}{M_n A_{m-1} M_n}$$

by our quoted inequality and consequently we have conclusion (ii).

Theorem 53. *If $a_n \geqq a_{n+1}$ a necessary and sufficient condition that $\sum_{n=1}^{\infty} a_n$ converge is that $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converge.*

PROOF: Let $s_n = \sum_{n=1}^{n} a_n$ and $t_k = \sum_{k=1}^{k} 2^k a_{2^k}$. Then

$$s_{2^{k+1}-1} = a_1 + (a_2 + a_3) + \ldots + (a_{2^k} + \ldots + a_{2^{k+1}-1})$$
$$\leqq a_1 + 2a_2 + \ldots + 2^k a_{2^k} = t_k + a_1,$$

and consequently, when $n \leqq 2^{k+1} - 1$,

(2) $$s_n \leqq t_k + a_1.$$

Moreover,

$$s_{2^{k+1}} = a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots + (a_{2^k+1} + \ldots + a_{2^{k+1}})$$
$$\geqq a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^k a_{2^{k+1}} = a_1 + \tfrac{1}{2} t_{k+1}.$$

Consequently, when $n \geqq 2^{k+1}$, that is $k \leqq \frac{\log n}{\log 2} - 1$

(3) $$t_{k+1} \leqq 2 s_n - 2 a_1.$$

Now by Theorem 8, in case $s_n \to S$, $t_k \leqq 2S - 2a_1$ and by Theorem 17 t_k approaches a limit. Likewise in case $t_n \to T$, by (2) $s_n \leqq T + a_1$ and hence by Theorem 17 s_n approaches a limit.

A generalization of this theorem is the following.

Theorem 54. *Given $a_n \geqq a_{n+1} > 0$, also that g_k is a sequence of positive integers so chosen that $g_k > g_{k-1}$ and*

$$g_{k+1} - g_k \leqq M(g_k - g_{k-1})$$

where $M > 0$ is fixed; then a necessary and sufficient condition that $\sum_{n=1}^{\infty} a_n$ converge is that $\sum_{k=1}^{\infty} (g_{k+1} - g_k) a_{g_k}$ converge.

PROOF: Let $s_n = \sum_{n=1}^{n} a_n$ and $t_k = \sum_{k=1}^{k} (g_{k+1} - g_k) a_{g_k}$. Also let $A = \sum_{n=1}^{g_1 - 1} a_n$. Then

$$s_{g_k} - 1 = A + (a_{g_1} + \dots + a_{g_2 - 1}) + \dots + (a_{g_{k-1}} + \dots + a_{g_k - 1})$$
$$\leqq A + (g_2 - g_1) a_{g_1} + \dots + (g_k - g_{k-1}) a_{g_{k-1}} = A + t_{k-1}.$$

Likewise

$$s_{g_k} = A + a_{g_1} + (a_{g_1+1} + \dots + a_{g_2}) + \dots$$
$$+ (a_{g_{k-1}+1} + \dots + a_{g_k}) \geqq A + a_{g_1} + (g_2 - g_1) a_{g_2} + \dots$$
$$+ (g_k - g_{k-1}) a_{g_k} \geqq A + a_{g_1} + \frac{1}{M}(g_3 - g_2) a_{g_2} + \dots$$
$$+ \frac{1}{M}(g_{k+1} - g_k) a_{g_k}$$
$$= A + a_{g_1}\left(1 - \frac{1}{M}\right) + \frac{1}{M} t_k.$$

We now have relations corresponding to (2) and (3) under the previous theorem, from which point the reasoning is a repetition of that employed there.

Theorem 55. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ *converges.* CONCLUSION:

$\sum_{n=1}^{\infty} \frac{a_n}{r_n^{\alpha}}$, *where* $r_n = \sum_{\gamma = n}^{\infty} a_{\gamma}$, *converges if* $\alpha < 1$ *and diverges if* $\alpha \geqq 1$.

PROOF: Suppose $\alpha = 1$, then

$$\frac{a_n}{r_n} + \frac{a_{n+1}}{r_{n+1}} + \dots + \frac{a_{n+k}}{r_{n+k}} \geq \frac{a_n + a_{n+1} + \dots + a_{n+k}}{r_n} = 1 - \frac{r_{n+k+1}}{r_n}.$$

For any fixed n, k can be taken so large that

$$\frac{r_{n+k+1}}{r_n} < \frac{1}{2};$$

consequently k can be taken so large that

$$\frac{a_n}{r_n} + \frac{a_{n+1}}{r_{n+1}} + \dots + \frac{a_{n+k}}{r_{n+k}} \geqq \frac{1}{2}$$

which, according to Theorem 21, assures us of divergence.

In case $\alpha > 1$, when $r_n < 1$ $\frac{a_n}{r_n^{\alpha}} > \frac{a_n}{r_n}$, and consequently divergence is assured.

Suppose $\alpha < 1$. Let $q_n = \frac{r_{n+1}}{r_n}$. Choose a fixed number τ so that $0 < \tau < 1$ and $\alpha < 1 - \tau$. Then when $r_n < 1$, that is when n is large enough,

$$\frac{a_n}{r_n^{\alpha}} < \frac{a_n}{r_n^{1-\tau}} = \frac{r_n - r_{n+1}}{r_n^{1-\tau}} = \frac{1 - q_n}{1 - q_n^{\tau}} (r_n^{\tau} - r_{n+1}^{\tau}) < \frac{1}{\tau} (r_n^{\tau} - r_{n+1}^{\tau}),$$

since
$$\frac{1 - q_n}{1 - q_n^{\tau}} < \frac{1}{\tau}.$$

But, $\frac{1}{\tau}(r_n^{\tau} - r_{n+1}^{\tau})$ is the general term of a convergent series and the proof is completed by Theorem 36.

Theorem 56. *Given that $a_n > a_{n+1}$ and that a function $f(x)$ integrable over any finite interval and monotonically decreasing can be found such that $f(n) = a_n$. Then, $\sum_{n=1}^{\infty} a_n$ converges in case there exists a fixed x_0 such that*

$$\frac{e^x f(e^x)}{f(x)} \leqq \theta < 1$$

when $x > x_0$, and diverges in case $\frac{e^x f(e^x)}{f(x)} \geqq 1$ when $x > x_0$.

PROOF: Suppose $\dfrac{e^x f(e^x)}{f(x)} \leqq \theta < 1$ when $x > x_0$, then

$$\int_{e^{x_0}}^{e^x} f(\tau)\,d\tau = \int_{x_0}^{x} e^t f(e^t)\,dt \leqq \theta \int_{x_0}^{x} f(t)\,dt$$

and hence

$$(1-\theta)\int_{e^{x_0}}^{e^x} f(t)\,dt \leqq \theta\left[\int_{x_0}^{x} f(t)\,dt - \int_{e^{x_0}}^{e^x} f(t)\,dt\right]$$

$$\leqq \theta\left[\int_{x_0}^{e^x} f(t)\,dt - \int_{e^{x_0}}^{e^x} f(t)\,dt\right] \leqq \theta \int_{x_0}^{e^{x_0}} f(t)\,dt.$$

But this last integral is between constant limits and hence is a constant. Hence $\int_{e^{x_0}}^{e^x} f(t)\,dt$ is less than some fixed number. Hence $\int_{x_0}^{x} f(t)\,dt$ is also, and consequently the series converges by an analogue of Theorem 17 and by Theorem 34.

Suppose, however, that $\dfrac{e^x f(e^x)}{f(x)} \geqq 1$ when $x > x_0 > 1$. Then, assuming $x > e^{x_0}$,

$$\int_{x_0}^{e^x} f(t)\,dt \geqq \int_{e^{x_0}}^{e^x} f(t)\,dt = \int_{x_0}^{x} e^t f(e^t)\,dt \geqq \int_{x_0}^{x} f(t)\,dt \geqq \int_{x_0}^{e^{x_0}} f(t)\,dt,$$

a positive constant. Hence $\sum_{n=x}^{e^x} a_n \not\rightarrow 0$ as $x \rightarrow \infty$, and consequently by Theorem 21 the series diverges.

EXERCISES

53 to 100. Investigate the convergence or divergence of series whose general terms are given below. In case a letter as a or b occurs in the formula, consider all possible cases.

$$\frac{n-1}{2n+3},\quad \frac{1}{n}\sin\frac{1}{n},\quad \frac{1}{n(n+1)},\quad \frac{n}{2^n},\quad \frac{n!}{n^5},\quad \frac{2^n}{n^{10}},\quad \frac{n^{10}}{10^n},$$

$$\frac{n^n}{n!},\quad \frac{1}{n^n},\quad \frac{1}{n2^n},\quad \frac{n!}{3.5.7\ldots(2n+1)},\quad \left(\frac{n!}{3.5.7\ldots(2n+1)}\right)^2,$$

$$\sin\frac{1}{n},\quad \sin^2\frac{1}{n},\quad \tan^{-1}\frac{1}{n},\quad \frac{1}{\sqrt{n}}\tan\frac{1}{n},\quad \frac{1}{n+\sqrt{n}},$$

$$\frac{1}{(n+1)^2-n^2},\quad \frac{\log n}{n^a},\quad \frac{1}{n(\log n)^a},\quad \cot^{-1} n,\quad \frac{n}{n^2+n-1},$$

$$\frac{1}{n^{1+\frac{1}{n}}},\quad \frac{n^a}{n!},\quad \frac{n+\sqrt{n}}{n^2-n},\quad \frac{(n!)^2}{(2n)!},\quad \frac{2^n(n!)}{n^n},\quad \frac{3^n(n!)}{n^n},$$

$$\frac{1}{1+a^n},\quad (\sqrt[n]{a}-1),\quad (\sqrt{n+1}-\sqrt{n}),\quad (\sqrt[3]{n+1}-\sqrt[3]{n}),$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{n},\quad \frac{1}{(\log n)^n},\quad \frac{1}{(\log\log n)^n},\quad \frac{1}{(\log\log n)^{\log n}},$$

$$\frac{1}{(\log\log\log n)^{\log n}},\quad a^{\sqrt{n}},\quad a^{\log n},\quad a^{\log\log n},\quad \left(1-\frac{\log n}{n}\right)^n,$$

$$\frac{1}{a^{a+\frac{b}{n}}},\quad \frac{(a+1)(2a+1)\ldots(na+1)}{(b+1)(2b+1)\ldots(nb+1)},\quad \frac{1}{2n+1}\cdot\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)},$$

$$\frac{a^n}{\sqrt[n]{n+n^n}},\quad \frac{1}{\log(1+n).\log(1+n^n)},\quad n^{-\log n},\quad n^{-\log\log n}.$$

101. Obtain an upper bound* for the remainder after n terms in as many of the convergent series of the last set as you can. Here particular attention is called to Theorems 34 and 36.

102. Is it necessary for convergence that $\lim\limits_{n\rightarrow\infty}\dfrac{a_{n+1}}{a_n}$ exist? Establish your conclusion by means of an example.

103. Is the following theorem correct or incorrect?

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \frac{a_{n+1}}{a_n} < \frac{\log n}{\log(n+1)}.$$

104. Is the following theorem correct or incorrect?

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \frac{a_{n+1}}{a_n} < \frac{n\log^2 n}{(n+1)\log^2(n+1)}.$$

105. Prove: If $\sum_{n=1}^{\infty} a_n$ is divergent, a second divergent series $\sum_{n=1}^{\infty} b_n$ can be found such that $\lim\limits_{n\rightarrow\infty}\dfrac{b_n}{a_n} = 0$.

* If $r_n < M$, M is called an upper bound for r_n.

When this is the case, $\sum_{n=1}^{\infty} b_n$ is sometimes said to diverge more slowly than $\sum_{n=1}^{\infty} a_n$. The theorem of the exercise states that no matter what divergent series of positive terms is considered a more slowly diverging series of positive terms exists.

106. Prove: If $\sum_{n=1}^{\infty} a_n$ converges, a second convergent series $\sum_{n=1}^{\infty} b_n$ exists such that $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=0$.

In other words, given any convergent series of positive terms there always exists a series of positive terms which converges more slowly. The existence of one such series implies the existence of an infinite number.

107. Prove: If $a_n > a_{n+1} \to 0$, a divergent series $\sum_{n=1}^{\infty} d_n$ exists such that $\sum_{n=1}^{\infty} a_n d_n$ converges.

108. Prove: If $a_n < a_{n+1} \to \infty$, a convergent series $\sum_{n=1}^{\infty} d_n$ exists such that $\sum_{n=1}^{\infty} a_n d_n$ diverges.

109. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ both diverge, what can be said of $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$, of $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}$, of $\sum_{n=1}^{\infty} \frac{a_n}{1+(a_n)^2}$?

110. If $\sum_{n=1}^{\infty} a_n$ converges, what can be said of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^{\frac{1}{2}+\delta}}, \quad \delta > 0\,?$$

111. Illustrate by figures at least five of the theorems of this chapter.

112. Prove: If

$$\frac{a_{n+1}}{a_n} = \frac{1}{1+\frac{\alpha_n}{n}} \quad \text{and} \quad \alpha_n = 1 + \frac{\beta_n}{n}$$

where β_n remains finite, then $\sum_{n=1}^{\infty} a_n$ is divergent.

113. Investigate, by a method of your own, the series $\sum_{n=1}^{\infty} a_n$, where $\frac{a_{n+1}}{a_n} = 1 + \frac{\alpha}{n} + \frac{\beta_m}{n^\mu}$, $\mu > 1$ and β_n remaining finite.

114. Prove: If $\frac{\log \frac{1}{a_n}}{\log n} \to l > 1$, $\sum_{n=1}^{\infty} a_n$ converges. If

$$\frac{\log \frac{1}{a_n}}{\log n} \to l < 1, \quad \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

115. Generalize the last theorem.

116. Prove: If for every n,

$$(n\log n)\left[(n \log n)\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \leqq K$$

(independent of n), then $\sum_{n=1}^{\infty} a_n$ is divergent.

117. Prove: For $\sum_{n=1}^{\infty} a_n$ to converge it is necessary that

$$n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \to \infty.$$

118. Prove, by a method of your own: If

$$(1-\sqrt[n]{a_n})\frac{n}{\log n} \geqq l > 1,$$

then $\sum_{n=1}^{\infty} a_n$ converges, and if

$$(1-\sqrt[n]{a_n})\frac{n}{\log n} \leqq 1$$

it diverges.

119. Prove: If

$$\frac{a_n}{a_{n+1}} = \frac{n^p + \alpha_1 n^{p-1} + \ldots + \alpha_p}{n^p + \beta_1 n^{p-1} + \ldots + \beta_p},$$

where $p > 0$ and $\alpha_1, \ldots \alpha_p$ and $\beta_1, \ldots \beta_p$ are independent of n, then $\sum_{n=1}^{\infty} a_n$ is convergent if $\alpha_1 - \beta_1 > 1$ and divergent if $\alpha_1 - \beta_1 \leqq 1$.

120. Show by examples that, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ may converge or diverge. Prove that, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges.

121. State all the theorems of these exercises, with hypotheses and conclusions.

CHAPTER V

SERIES SOME OF WHOSE TERMS ARE POSITIVE AND SOME NEGATIVE

In this chapter, as the heading indicates, we shall treat series whose terms are real but not necessarily all of the same sign. The reality of the terms will be assumed and not specifically mentioned in the various hypotheses.

First let us call attention to Theorem 26 which has been proved for series where the terms are real or complex. No theorem is more fundamental in the present chapter.

Definition 12. *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges,* $\sum_{n=1}^{\infty} a_n$ *is said to converge absolutely or to be absolutely convergent.*

Definition 13. If $\sum_{n=1}^{\infty} |a_n|$ *diverges but* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *is said to converge conditionally or non-absolutely.*

All the theorems of Chapter IV could now be restated for absolutely convergent series. The changes in wording would be trivial. However, this is unnecessary as the natural course of procedure in any investigation, to which these theorems can be applied, is to begin by forming the series of absolute values, and then to apply established theorems to it.

Let the series to be considered be $\sum_{n=0}^{\infty} a_n$, and let, as usual, $s_n = a_0 + \ldots + a_{n-1}$. Denote the positive terms in the order of their occurrence by $b_0, b_1, \ldots$, and the negative terms by $-c_0, -c_1, \ldots$.

Let $\sigma_n = b_0 + \ldots + b_{n-1}$ and $\tau_n = c_0 + \ldots + c_{n-1}$. We assume an infinite number of positive and an infinite number of negative terms. There is no loss of generality in this if the

question at issue is one of convergence or divergence. If there are, for example, only a finite number of negative terms the last of which is a_k, we simply examine the series

$$\sum_{n=k+1}^{\infty} a_n.$$

Theorem 57. HYPOTHESIS: $\sum_{n=0}^{\infty} a_n$ *converges.* CONCLUSION: *Either* $\sum_{n=0}^{\infty} b_n$ *and* $\sum_{n=0}^{\infty} c_n$ *both converge or both diverge.*

PROOF: Suppose the contrary and, for example, that $\sum_{n=0}^{\infty} b_n$ diverges and $\sum_{n=0}^{\infty} c_n$ converges. $s_n = \sigma_m - \tau_\mu$, where $m+\mu = n$. As $n \to \infty$, $\mu \to \infty$, and $m \to \infty$, $s_n \to$ a limit and $\tau_\mu \to$ a limit. Hence $\sigma_m = s_n + \tau_\mu \to$ a limit which is contrary to the hypothesis.

Examples illustrative of each case are readily set up.

Theorem 58. HYPOTHESIS: $\sum_{n=1}^{\infty} b_n$ *and* $\sum_{n=1}^{\infty} c_n$ *both converge.* CONCLUSION: $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

PROOF: Let $s_n = \sigma_m - \tau_\mu$ as above, and let

$$S_n = |a_1| + \ldots + |a_n|.$$

Then, $S_n = \sigma_m + \tau_\mu$ approaches a limit since σ_m and τ_μ approach limits.

Theorem 59. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

CONCLUSION: $\sum_{n=0}^{\infty} b_n$ *and* $\sum_{n=0}^{\infty} c_n$ *both converge.*

PROOF: Let $\sum_{n=0}^{\infty} |a_n| = S.$ $\sigma_n \leqq S$ and $\tau_n \leqq S$. Hence, by Theorem 17, we draw the desired conclusion.

Theorem 60. HYPOTHESES: (i) *The terms of* $\sum_{n=0}^{\infty} a_n$ *are alternately plus and minus*; (ii) $|a_{n+1}| \leqq |a_n|$; (iii) $a_n \to 0$.

CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges.*

PROOF: Suppose $a_0 > 0$ and let $a_n = (-1)^n b_n$.

$$s_{2k} = (b_0 - b_1) + (b_2 - b_3) + \ldots + (b_{2k-2} - b_{2k-1})$$

is positive by (ii). Moreover,

$$s_{2k} = b_0 - (b_1 - b_2) - \ldots - (b_{2k-3} - b_{2k-2}) - b_{2k-1},$$

and consequently $s_{2k} < b_0$.

Moreover, s_{2k} does not decrease as k increases. Consequently s_{2k} approaches a limit by Theorem 17.

$$s_{2k+1} = s_{2k} + b_{2k}.$$

But $b_{2k} \to 0$, and hence s_{2k+1} approaches the same limit as s_{2k}; that is s_n approaches a limit, n passing through all values, as was to be proved.

Corollary. HYPOTHESES *are the same as in the theorem.* CONCLUSION: $|r_n| < |a_n|$.

PROOF: $|r_n| = |a_n - (a_{n+1} - a_{n+2}) - \ldots|$. The conclusion is immediate.

EXERCISES

122 to 127. For what real values of x do the series whose general terms are given below converge; converge absolutely?

$$(-1)^n \frac{1}{n}(x^2)^n, \quad (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-1)^n \frac{1}{n!}(x^2)^n,$$

$$(-1)^n \frac{1}{x+n}, \quad (-1)^n \frac{2^n}{x+n}, \quad (-1)^n \frac{1}{n^p}.$$

128. Give an upper limit for the remainder after n terms in each of the above series when convergent.

129. Discuss convergence of

$$\frac{1}{x} + \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4} - \frac{1}{x+5} + \ldots$$

and

$$\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} - \frac{1}{x+5} + \ldots.$$

130. If $\epsilon_n = \begin{cases} 1 & \text{when } 2^{2k} \leqq n < 2^{2k+1} \\ -1 & \text{when } 2^{2k+1} \leqq n < 2^{2k+2}, \end{cases}$

discuss convergence of

$$\sum_{n=2}^{\infty} \frac{\epsilon_n}{n \log n}, \quad \sum_{n=1}^{\infty} \frac{\epsilon_n}{n}.$$

131. Prove that, if in $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ signs be changed so that p negative signs follow q positive ones throughout the series and $p \neq q$, the series remains divergent, but that if $p = q$ it converges.

132. Prove that, if $a > 0$,

$$a - a^{\frac{1}{2}} + a^{\frac{1}{3}} - a^{\frac{1}{4}} + a^{\frac{1}{5}} - a^{\frac{1}{6}} + \dots$$

diverges, but that it can be made to converge by the proper insertion of parentheses.

133. Prove that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ diverges if

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n},$$

although the terms are alternately positive and negative and approach zero.

134. Construct another example similar to that of the previous exercise.

135. Examine

$$1 - \frac{\alpha \,.\, \beta}{1 \,.\, \gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \,.\, 2 \,.\, \gamma(\gamma+1)} - \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \,.\, 2 \,.\, 3 \,.\, \gamma(\gamma+1)(\gamma+2)} + \dots$$

for convergence.

136. Examine

$$1 - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{4} + \frac{1}{5^2} - \frac{1}{6} + \frac{1}{7^2} - \frac{1}{8} + \dots$$

and $$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \dots$$

for convergence.

137. From the series of Exercises 53 to 100 pick out those series which diverge but which can be made to converge by the proper insertion of minus signs.

138. Draw figures to illustrate at least two theorems of this chapter.

139. Prove the following theorem: HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ converges but not absolutely. p_n denotes the number of positive terms in the first n terms and q_n the number of negative terms. CONCLUSION: $\frac{p_n}{q_n} \to 1$.

This theorem is most useful in testing particular series for convergence.

CHAPTER VI

SERIES WHOSE TERMS ARE COMPLEX

All theorems in Chapter III are for series whose terms are either real or complex. The theorems here are more particular and are supplementary to those given there.

The definitions of absolute convergence and conditional convergence are the same as those for series all of whose terms are real as given in the previous chapter. A most fundamental theorem here as there is number 26, namely that *a series is convergent if it is absolutely convergent, and that for such a series*

$$\left|\sum_{n=0}^{\infty} a_n\right| \leqq \sum_{n=0}^{\infty} |a_n|.$$

In an investigation for convergence we usually begin by investigating for absolute convergence. As has been pointed out, if it is desired to apply the tests of Chapter IV directly, any theorem of that chapter can be reworded for absolute convergence. However, many convergent series are not absolutely convergent and other methods frequently must be employed.

Let $a_n = \beta_n + \gamma_n i$, where β_n and γ_n are real.

Theorem 61. *A necessary and sufficient condition that* $\sum_{n=0}^{\infty} a_n$ *converge is that both* $\sum_{n=0}^{\infty} \beta_n$ *and* $\sum_{n=0}^{\infty} \gamma_n$ *converge.*

PROOF: Let $s_n = a_0 + \dots + a_{n-1}$, $\sigma_n = \beta_0 + \dots + \beta_{n-1}$, and $\tau_n = \gamma_0 + \dots + \gamma_{n-1}$. Then $s_n = \sigma_n + \tau_n i$. Apply Theorems 7 and 9, corollaries, and proof is complete.

Theorem 62. HYPOTHESIS: $\sum_{n=0}^{\infty} |\beta_n|$ and $\sum_{n=0}^{\infty} |\gamma_n|$ *converge to B and C respectively.* CONCLUSION: $\sum_{n=0}^{\infty} |a_n|$ *converges and* $\sum_{n=0}^{\infty} |a_n| \leqq B + C.$

PROOF: Let $S_n = |a_0| + \dots + |a_{n+1}|$, $T_n = |\gamma_0| + \dots + |\gamma_{n-1}|$ and $\Sigma_n = |\beta_0| + \dots + |\beta_{n-1}|$. Then $S_n \leqq \Sigma_n + T_n \leqq B + C$. From this relation we draw the desired conclusion by means of Theorem 8.

Theorem 63. HYPOTHESIS: $\sum_{n=0}^{\infty} |a_n|$ *converges to S.* CONCLUSION: $\sum_{n=0}^{\infty} |\beta_n|$ *and* $\sum_{n=0}^{\infty} |\gamma_n|$ *both converge and*

$$\sum_{n=0}^{\infty} |\beta_n| \leqq S \text{ and } \sum_{n=0}^{\infty} |\gamma_n| \leqq S.$$

PROOF: $|\beta_n| \leqq |a_n|$, $|\gamma_n| \leqq |a_n|$. Hence, using the notation of the previous theorem, $\Sigma_n \leqq S$ and $T_n \leqq S$. The theorem follows.

Theorem 64. HYPOTHESIS:

$$a_n = r_n e^{\phi_n i}, \quad |\phi_n - \phi_m| \leqq \pi - 2\delta, \quad \pi > 2\delta > 0,$$

n and m any integers. CONCLUSION: *Either* $\sum_{n=0}^{\infty} |a_n|$ *converges or* $|s_n| = \left|\sum_{n=0}^{n-1} a_n\right| \to \infty.$

PROOF: Multiply each term of the series through by $e^{\psi i} = \epsilon$, so choosing ψ that if

$$a_n\epsilon = r_n e^{\chi_n i}, \quad -\frac{\pi}{2} + \delta \leqq \chi_n \leqq \frac{\pi}{2} - \delta, \quad 0 < \delta < \frac{\pi}{2}.$$

This will not affect convergence or divergence as described in the theorem as ϵ is a constant different from zero. Let $b_n = a_n\epsilon$ and consider $\sum_{n=0}^{\infty} b_n$. Moreover let

$$\bar{s}_n = b_0 + \dots + b_{n-1} \text{ and } s_n = a_0 + \dots + a_{n-1}. \quad |\bar{s}_n| = |s_n|.$$

(1) Suppose $\sum_{n=0}^{\infty} b_n$ converges. Then, as

$$b_n = r_n(\cos\chi_n + i\sin\chi_n),$$

by Theorem 63 $\sum_{n=0}^{\infty} r_n \cos\chi_n$ converges. But

$$\cos\chi_n \geqq \cos\left(\frac{\pi}{2} - \delta\right) = \sin\delta.$$

Hence $\sum_{n=0}^{\infty} r_n \sin\delta$ converges. But $\sin\delta$ is a constant.

Hence $\sum_{n=0}^{\infty} r_n = \sum_{n=0}^{\infty} |a_n|$ converges.

(2) Suppose $\sum_{n=0}^{\infty} b_n$ diverges. Then, by Theorem 32, $\sum_{n=0}^{\infty} r_n$ diverges to ∞, that is $\sum_{n=0}^{n-1} r_n \to \infty$. Hence

$$\sum_{n=0}^{\infty} r_n \sin\delta \text{ and hence } \sum_{n=0}^{\infty} r_n \cos\chi_n$$

diverges to ∞. But $|s_n| \geqq \sum_{n=0}^{n-1} r_n \cos\chi_n$. Hence $|s_n| \to \infty$ as was to be proved.

Theorem 65. *Given* $\dfrac{a_{n+1}}{a_n} = 1 - \dfrac{\alpha}{n} - \dfrac{A_n}{n^\lambda}$, *where* $\alpha = \beta + i\gamma$ *is a constant,* $|A_n| < K$, *a constant, and* $\lambda > 1$; *then, if* $\beta > 1$ $\sum_{n=1}^{\infty}$ *converges absolutely, if* $\beta \leqq 1$ *it does not converge absolutely, if* $\beta \leqq 0$ *it diverges.*

PROOF: Assume $\beta > 1$. Then, choose $\beta' > 1$ so that $\beta - \beta' = \delta > 0$. Then

$$\left|1 - \frac{\alpha}{n} - \frac{A_n}{n^\lambda}\right| < \left|1 - \frac{\beta + i\gamma}{n}\right| + \frac{K}{n^\lambda}$$

$$= \sqrt{\left(1 - \frac{\beta}{n}\right)^2 + \frac{\gamma^2}{n^2}} + \frac{K}{n^\lambda}$$

$$= \sqrt{\left(1 - \frac{\beta'}{n}\right)^2 - 2\left(1 - \frac{\beta'}{n}\right)\frac{\delta}{n} + \frac{\delta^2}{n^2} + \frac{\gamma^2}{n^2}} + \frac{K}{n^\lambda} < 1 - \frac{\beta'}{n},$$

for sufficiently large values of n, since $2\left(1 - \dfrac{\beta'}{n}\right)\dfrac{\delta}{n}$ is an in-

finitesimal of lower order than $\dfrac{\delta^2}{n^2}$, &c., when $n \to \infty$; that is, $\left|\dfrac{a_{n+1}}{a_n}\right| < 1 - \dfrac{\beta'}{n}$ for n sufficiently great. From which

$$\left|\frac{a_n}{a_{n+1}}\right| > 1 + \frac{\beta'}{n - \beta'}.$$

Now apply Theorem 42 and absolute convergence is established.

Next assume $0 < \beta \leqq 1$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| \geqq \left|1 - \frac{\beta + i\gamma}{n}\right| - \left|\frac{A_n}{n^\lambda}\right| \geqq 1 - \frac{\beta}{n} - \frac{|A_n|}{n^\lambda} \geqq 1 - \frac{\beta}{n}.$$

Apply Theorem 42 and divergence of $\sum_{n=1}^{\infty} |a_n|$ is readily proved.

Assume $\beta = 0$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|1 - \frac{\gamma i}{n} - \frac{A'_n + B'_n i}{n^\lambda}\right|$$

$$= \sqrt{\left(1 - \frac{A'_n}{n^\lambda}\right)^2 + \left(\frac{\gamma}{n} + \frac{B'_n}{n^\lambda}\right)^2} \geqq 1 - \frac{A'_n}{n^\lambda} \geqq 1 - \frac{K}{n^\lambda}.$$

When m is great enough $1 - \dfrac{K}{m^\lambda} > 0$. Hence, by multiplication,

$$\left|\frac{a_n}{a_m}\right| = \left|\frac{a_{m+1}}{a_m}\right| \cdot \left|\frac{a_{m+2}}{a_{m+1}}\right| \cdots \left|\frac{a_n}{a_{n-1}}\right|$$

$$> \left(1 - \frac{K}{m^\lambda}\right)\left(1 - \frac{K}{(m+1)^\lambda}\right) \cdots \left(1 - \frac{K}{n^\lambda}\right).$$

But

$$\log\left(1 - \frac{K}{m^\lambda}\right)\left(1 - \frac{K}{(m+1)^\lambda}\right) \cdots \left(1 - \frac{K}{n^\lambda}\right)$$

$$= \sum_{\nu=m}^{n} \log\left(1 - \frac{K}{\nu^\lambda}\right) > \sum_{\nu=m}^{\infty} \log\left(1 - \frac{K}{\nu^\lambda}\right),$$

since $1 - \dfrac{K}{\nu^\lambda} < 1$.

But this series converges, [redacted] is readily shown by Theorem 38 and Example 3, page 30. This means that

$$\log\left|\frac{a_n}{a_m}\right| > c_1;$$

that is that $\left|\dfrac{a_n}{a_m}\right| > c_2$ or $|a_n| > c_2|a_m|$.

Hence, $a_n \not\to 0$, and divergence is established.

Assume $\beta < 0$. As when $0 < \beta \leqq 1$,

$$\left|\frac{a_{n+1}}{a_n}\right| \geqq 1 - \frac{\beta}{n} - \frac{|A_n|}{n^\lambda},$$

from which we see that for sufficiently great values of n,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1.$$

This means $a_n \not\to 0$, and divergence follows.

Theorem 66. HYPOTHESES: (i) $\dfrac{a_{n+1}}{a_n} = 1 - \dfrac{\alpha}{n} - \dfrac{A_n}{n^\lambda}$ *as in the previous theorem*; (ii) $\beta > 0$. CONCLUSIONS:

$$\sum_{n=0}^{\infty}(-1)^n a_n \text{ and } \sum_{n=0}^{\infty}|a_n - a_{n+1}|$$

both converge.

PROOF: As shown in the previous proof, when $\beta > 1$, if we take $0 < \beta' < \beta$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| < 1 - \frac{\beta'}{n};$$

and hence, by multiplication

$$\left|\frac{a_n}{a_m}\right| < \left(1 - \frac{\beta'}{m}\right)\left(1 - \frac{\beta'}{m+1}\right)\cdots\left(1 - \frac{\beta'}{n-1}\right).$$

When $n \to \infty$ the right-hand member approaches zero, as is readily proved by taking its logarithm, allowing n to become infinite and showing that the resulting infinite series diverges to $-\infty$. Details are omitted. It results that

$a_n \to 0$. Consequently, $\sum_{n=0}[|a_n| - |a_{n+1}|]$ converges. Now

$$\frac{|a_n - a_{n+1}|}{|a_n| - |a_{n+1}|} = \frac{\left|1 - \dfrac{a_{n+1}}{a_n}\right|}{1 - \left|\dfrac{a_{n+1}}{a_n}\right|} \leqq \frac{\left|\dfrac{\alpha}{n} + \dfrac{A_n}{n^\lambda}\right|}{\dfrac{\beta'}{n}} \to \frac{|\alpha|}{\beta'}.$$

Hence, by Theorem 38, $\sum_{n=0}^{\infty}|a_n - a_{n+1}|$ converges.

We next examine $\sum_{n=0}^{\infty}(-1)^n a_n$.

$\sum_{k=0}|a_{2k} - a_{2k+1}|$ converges as it is a series of positive terms composed of every other term of $\sum_{n=0}^{\infty}|a_n - a_{n+1}|$ which we have just proved convergent. Hence, by theorem 26,

$$\sum_{k=0}^{\infty}(a_{2k} - a_{2k+1})$$

converges. But, as $a_n \to 0$, we can drop the parentheses and have the desired result; namely, that $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

EXERCISES

140. If $\sum_{n=1}^{\infty} a_n$ converges and $am\,a_n$ denotes the angle of a_n in the complex plane, what can you say as to $\sum_{n=1}^{\infty} am\,a_n$? Conversely, if $\sum_{n=1}^{\infty} am\,a_n$ converges, what can you say of

$$\sum_{n=1}^{\infty} a_n?$$

141. Discuss convergence of $\sum_{n=1}^{\infty}\dfrac{(-1)^{\sqrt{n}}}{n}$.

142–143. Show that

$$\sum_{n=0}^{\infty}\left[\left(z+n+\frac{1}{2}\right)\log\left(1+\frac{1}{z+n}\right)-1\right]$$

and

$$\sum_{n=0}^{\infty}\left(\frac{1}{z+n}-\log\left(1+\frac{1}{z+n}\right)\right)$$

both converge for all values of z except $0, -1, -2, -3, \dots$. The determination of the logarithm is, in each instance, to be such that its imaginary part is numerically as small as possible.

144. Discuss convergence of

$$\frac{1}{z}+\frac{1}{z+1}-\frac{1}{z+2}+\frac{1}{z+3}+\frac{1}{z+4}-\frac{1}{z+5}+\dots$$

for complex values of z.

145. Examine for absolute convergence,

$$\sum_{n=1}^{\infty}\left[\frac{1}{z-c_n}+\frac{1}{c_n}+\frac{z}{c_n^2}+\dots+\frac{z^{n-1}}{c_n^n}\right],$$

where $|c_n|\to\infty$ and $z\neq c_n$ for any n.

146–150. Examine the following series for convergence at all points on the unit circle:

$$\sum_{n=1}^{\infty}\frac{z^n}{n},\quad \sum_{k=1}^{\infty}\frac{z^{2k-1}}{2k-1},\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n},\quad \sum_{n=2}^{\infty}\frac{z^n}{(n-1)(n+1)},$$

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{z^{2n}}{n}.$$

151. Prove, by a method of your own: If

$$\frac{a_n}{a_{n+1}}=1+\frac{\alpha+\beta i}{n}+\frac{\gamma}{n^{\lambda}},$$

where γ remains finite and $\lambda>1$; then, $\sum\limits_{n=1}^{\infty}a_n$ is absolutely convergent if $\alpha>1$ and does not converge absolutely if $\alpha\leqq 1$ and diverges if $\alpha\leqq 0$.

152. Prove, by a method of your own: If $\dfrac{a_n}{a_{n+1}}$ is of the form of the last example and $\alpha>0$, $\sum\limits_{n=1}^{\infty}|a_n-a_{n+1}|$ converges.

CHAPTER VII

TRANSFORMATIONS OF SERIES AND OPERATIONS WITH SERIES

In this chapter the terms of the series are any complex numbers unless a specific statement to the contrary is made.

Infinite series are not polynomials, and theorems holding for polynomials do not necessarily hold for infinite series. This fact will be vividly illustrated in the following theorems.

§ 1. Series formed from a given series.

Theorem 67. HYPOTHESIS: $\sum\limits_{n=0}^{\infty}a_n=s$ *converges absolutely.*

CONCLUSION: $\sum\limits_{n=0}^{\infty}a_{\lambda_n}$ *converges to* s *also, where* $\lambda_1, \lambda_2, \dots$ *are the integers* 1, 2, …, *chosen in any order, none being omitted.*

PROOF: Let $a_0+\dots+a_{n-1}=s_n$ and $a_{\lambda_1}+\dots+a_{\lambda_m}=\sigma_m$. Let m be given and choose n as great as possible so that each one of $a_1, \dots, a_n$ occurs among $a_{\lambda_1}, \dots, a_{\lambda_m}$. Denote those terms of $a_{\lambda_1}, \dots, a_{\lambda_m}$ not occurring among $a_1, \dots, a_n$ by $a_{\alpha_1}, \dots, a_{\alpha_{m-n}}$, $\alpha_1<\alpha_2<\dots<\alpha_{m-n}$. Then

$$\sigma_m-s=s_n-s+\sigma_m-s_n=s_n-s+a_{\alpha_1}+a_{\alpha_2}+\dots+a_{\alpha_{m-n}}.$$

Hence,

$$|\sigma_m-s|\leqq|s_n-s|+|a_{\alpha_1}|+|a_{\alpha_2}|+\dots$$
$$+|a_{\alpha_{m-n}}|\leqq|s_n-s|+[|a_n|+|a_{n+1}|+\dots].$$

Due to the convergence of the series, $\sum\limits_{n=0}^{\infty}a_n$, $|s_n-s|\to 0$ when $n\to\infty$, and due to the convergence of $\sum\limits_{n=0}^{\infty}|a_n|$,

$$[|a_n|+|a_{n+1}|+\dots]\to 0 \text{ also.}$$

Now, when $m\to\infty$, $n\to\infty$. Consequently, $|\sigma_m-s|\to 0$ when $m\to\infty$, which proves the theorem.

Theorem 68. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n$ *converges to* s; (ii) $\lambda_1, \lambda_2, \ldots$, *are the integers* 1, 2, ..., *chosen in any order, none being omitted*; (iii) $|n-\lambda_n| < G$, *a fixed number.*

CONCLUSION: $\sum_{n=0}^{\infty} a_{\lambda_n}$ *converges to* s.

PROOF: Use the notation of the previous theorem. Then take $m \leqq n+G$. Let $\epsilon > 0$ be given.

$$|\sigma_m - s| \leqq |s_n - s| + |a_{\alpha_1}| + \ldots + |a_{\alpha_{m-n}}|.$$

Let b be the largest of any of the terms $|a_{\alpha_1}|, \ldots, |a_{\alpha_{m-n}}|$. Then $|a_{\alpha_1}| + \ldots + |a_{\alpha_{m-n}}| \leqq Gb$. We know that $s_n \to s$. Consequently we can choose an $\bar{n}$; so that when $n > \bar{n}$,

$$|s_n - s| < \frac{\epsilon}{2}.$$

Now $a_n \to 0$; and since $\alpha_1 \to \infty$, $b \to 0$. Hence we can find an $\overline{m}$ such that, when $m > \overline{m}$, $bG < \frac{\epsilon}{2}$. Then, when both $m > \overline{m}$ and $n > \bar{n}$, which is true if $\overline{m} \geqq \bar{n} + G$ and $m > \overline{m}$,

$$|\sigma_n - s| < \epsilon,$$

which gives us the proof.

The last two theorems can be summarized by saying: The terms of an absolutely convergent series can be rearranged at pleasure without altering the sum; the terms of any convergent series can be rearranged at pleasure without altering the sum provided the displacement of no term exceeds some fixed number.

Theorem 69. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ *is a conditionally convergent series of real numbers, and* s *is any arbitrary real number.* CONCLUSION: *It is possible to choose the integers* 1, 2, 3, ... *in an order* $\lambda_1, \lambda_2, \lambda_3, \ldots$, *omitting none, so that*

$$\sum_{n=1}^{\infty} a_{\lambda_n} = s.$$

PROOF: Denote the positive terms in $\sum_{n=1}^{\infty} a_n$ by $b_1, b_2, \ldots$, and the negative terms by $-c_1, -c_2, \ldots$. The series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both diverge; otherwise, by Theorems 57 and 58, $\sum_{n=1}^{\infty} a_n$ would converge absolutely. Being series of positive terms, they diverge to plus infinity,* by Theorem 32.

Suppose $s > 0$ and begin by choosing the a_{λ_n}'s as b's until we have a sum greater than or equal to s. As soon as this occurs choose $-c$'s until the sum is less than s. As soon as this happens choose b's until the sum is greater than or equal to s, then $-c$'s until it is less than s, and so on indefinitely.

Denoting the partial sum thus formed by s_n, we have

$$|s - s_n| \leqq |a_j|,$$

where a_j is that term last added when s_n changed from being smaller than s to being greater than or equal to it, or from being greater than or equal to it to being smaller, $a_j \to 0$. Hence $s_n \to s$.

The arrangement is not unique. The details of the process can be changed in various ways.

In case s is negative or zero the process that we have described contemplates beginning with $-c$'s and proceeding as when s is positive.

Corollary 1. *In the rearrangment of the terms contemplated in this theorem a finite number of terms can be omitted*; since the resulting series with the terms taken in the original order would still be conditionally convergent.

Corollary 2. *The numbers* λ_n *can be so chosen, no positive integer being omitted, that* $\sum_{n=1}^{\infty} a_{\lambda_n}$ *will diverge to plus*

* It should be remembered that this is only a formal way of describing the behaviour of the sum of the first n terms when $n \to \infty$.

infinity, minus infinity, or so that* $\sum_{n=1}^{\infty} a_{\lambda_n}$ *will oscillate indefinitely over any prescribed interval.*

Proof follows closely that of the theorem and details are omitted.

Corollary 3. *In case a_n is complex, $a_n = \beta_n + i\gamma_n$, either $\sum_{n=1}^{\infty} \beta_n$ or $\sum_{n=1}^{\infty} \gamma_n$ must converge conditionally. If, for example, $\sum_{n=1}^{\infty} \beta_n$ is conditionally convergent the terms of the given series can be reordered so as to make $\sum_{n=1}^{\infty} \beta_n$ behave in a manner prescribed as in the theorem or as in Corollary 2.*

Theorem 70. HYPOTHESES: (i) $a_n > 0$; (ii) $a_n \to 0$; (iii) $\sum_{n=0}^{\infty} a_n$ *diverges*; (iv) t *is any arbitrary positive number.* CONCLUSION: *There exists a sequence of integers $\psi(n) \geqq n+1$ such that*

$$\lim_{n\to\infty} \sum_{\nu=n+1}^{\psi(n)} a_\nu = t.$$

PROOF: Suppose that when $n > m$, $a_n < t$. Let $n > m$ and choose $\psi(n)$ so that

$$a_{n+1} + \ldots + a_{\psi(n)-1} \leqq t \leqq a_{n+1} + \ldots + a_{\psi(n)}.$$

Since $a_n \to 0$,

$$(a_{n+1} + \ldots + a_{\psi(n)}) \to t.$$

The choice of $\psi(n)$ is not unique.

Theorem 71. HYPOTHESES: (i) $\sum_{n=0}^{\infty} (p_n + iq_n)$ *diverges*; (ii) $p_n > 0$; (iii) $p_n \to 0$; (iv) $\frac{q_n}{p_n} \to 0$; (v) t *is any arbitrary positive number.* CONCLUSION: *There exists a sequence of integers $\psi(n) \geqq n+1$ such that*

$$\lim_{n\to\infty} \sum_{\nu=n+1}^{\psi(n)} (p_\nu + iq_\nu) = t.$$

* See comment on Theorem 32 and the last footnote.

PROOF: Let $\epsilon > 0$ be given and let $\delta > 0$ be arbitrary. There exists an N, such that when $\nu \geqq N$, $|q_\nu| < \delta p_\nu$ and consequently, such that

$$\left| \sum_{\nu=n+1}^{\psi(n)} q_\nu \right| < \delta \sum_{\nu=n+1}^{\psi(n)} p_\nu.$$

Choose $\psi(n)$ such that, when $n > \overline{N} \geqq N$,

$$\left| \sum_{\nu=n+1}^{\psi(n)} p_\nu - t \right| < \delta.$$

We know from the previous theorem that this can be done. Now,

$$\left| \sum_{\nu=n+1}^{\psi(n)} (p_\nu + iq_\nu) - t \right| \leqq \left| \sum_{\nu=n+1}^{\psi(n)} p_\nu - t \right| + \left| \sum_{\nu=n+1}^{\psi(n)} iq_\nu \right|$$
$$\leqq \delta + \delta(t+\delta) < \epsilon$$

if $\delta < \frac{\epsilon}{1+2t}$ and $\delta < t$. Since δ is arbitrary but for being positive, the theorem is proved.

Theorem 72. HYPOTHESES: (i) $a_0 + a_1 + a_2 + \ldots$ *converges to s*; (ii)

$$a_0 + \ldots + a_k = \alpha_1,$$
$$a_{k+1} + \ldots + a_l = \alpha_2,$$
$$a_{l+1} + \ldots + a_m = \alpha_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where k, l, m, ... are any positive integers. CONCLUSION: $\sum_{n=1}^{\infty} \alpha_n$ *converges to s.*

PROOF: Let $\epsilon > 0$ be given. Let $s_n = a_0 + \ldots + a_{n-1}$ and $S_m = \alpha_1 + \ldots + \alpha_m$. Choose M so large that $|s_n - s| < \epsilon$ when $n > M$. Now, $S_m = s_{m'}$, where $m' \geqq m$. Consequently, when $m > M$, $|S_m - s| < \epsilon$; which constitutes proof.

We can loosely express the content of this theorem by saying: Parentheses can be inserted in a convergent series at pleasure without affecting its sum.

In the next theorem we shall continue the notation used in the theorem just proved.

Theorem 73. HYPOTHESES: (i) $\sum_{m=1}^{\infty} \alpha_m$ *converges to* s; (ii) $\alpha_m = a_k + \ldots + a_l$; (iii) $l - k < G$, *a fixed number*; (iv) $a_n \to 0$. CONCLUSION: $\sum_{n=0}^{\infty} a_n$ *converges to* s.

PROOF: Let $\epsilon > 0$ be given. $s_n = S_m + p_n$, where

$$p_n = \sum_{k=q}^{t} a_k,\ t-q < G.$$

Take M so large that when $m > M$, $|S_m - s| < \frac{\epsilon}{2}$. Then take $N \geqq GM$, so large that when $n > N$, $|a_n| < \frac{\epsilon}{2G}$. Then, when $n > N$, $m > M$ also; and consequently

$$|s_n - s| \leqq |S_m - s| + |p_n| < \epsilon.$$

Theorem 74. HYPOTHESES: $A_1 = a_1^{(1)} + a_2^{(1)} + \ldots$, $A_2 = a_1^{(2)} + a_2^{(2)} + \ldots$, $\ldots$, $A_k = a_1^{(k)} + a_2^{(k)} + \ldots$ *are convergent series.* CONCLUSION:

$$\sum_{n=1}^{\infty} (a_n^{(1)} + \ldots + a_n^{(k)}) = \sum_{n=1}^{\infty} \alpha_n = A_1 + \ldots + A_k.$$

PROOF: Let $s_n^{(m)} = a_1^{(m)} + \ldots + a_n^{(m)}$, $m = 1, \ldots, k$, and let $S_n = \alpha_1 + \ldots + \alpha_n$. Then $S_n = s_n^{(1)} + \ldots + s_n^{(k)}$. Hence, by Theorem 11, Cor., $S_n \to A_1 + \ldots + A_k$.

In the theorem just proved it has been shown that any fixed number of convergent series can be added term by term. The question of passing to an infinite number is answered in part by the following theorem, also see Theorem 90.

Theorem 75. HYPOTHESES: (i) $\sum_{k=0}^{\infty} A_k$ *converges, where* $A_k = \sum_{n=0}^{\infty} a_{k,n}$; (ii) *letting* $\sum_{n=P}^{\infty} a_{k,n} = {}_P r_k$, $\sum_{k=0}^{\infty} {}_P r_k = R_P$ *converges for all values of* P; (iii) $R_P \to 0$. CONCLUSION: $\sum_{k=0}^{\infty} a_{k,n}$ *converges to a value which we call* B_n *and*

$$\sum_{n=0}^{\infty} B_n = \sum_{k=0}^{\infty} A_k.$$

PROOF: $a_{k,n} = {}_n r_k - {}_{n+1} r_k$ and consequently $\sum_{k=0}^{\infty} a_{k,n}$ converges by hypothesis (ii). We write $A_k = A_k^P + {}_P r_k$. From which

$$\sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} A_k^P + \sum_{k=0}^{\infty} {}_P r_k = \sum_{n=0}^{P-1} B_n + R_P$$

by Theorem 74. But $R_P \to 0$. The theorem follows.

Theorem 76 (*Lemma*). HYPOTHESES: (i) $a_{n,p} \to 0$ *for every fixed* $p \geqq 0$; (ii) $\sum_{p=0}^{n} |a_{n,p}| < K$ *for every value of* $n > 0$; (iii) $x_n \to 0$. CONCLUSION:

$$\sigma_n = a_{n,0}x_0 + a_{n,1}x_1 + \ldots + a_{n,n}x_n \to 0.$$

PROOF: Let $\epsilon > 0$ be given, then if M is large enough, when $n > M$, $|x_n| < \frac{\epsilon}{2K}$, and as a result

$$|\sigma_n| < |a_{n,0}x_0 + \ldots + a_{n,M}x_M| + \frac{\epsilon}{2}.$$

Now with M held fast, choose n so large that

$$|a_{n,j}x_j| < \frac{\epsilon}{2M}$$

for $j = 0, \ldots, M$ simultaneously. Then $|\sigma_n| < \epsilon$, and the lemma is proved.

Let $\Delta^{(0)} a_k = a_k$, $\Delta^{(1)} a_k = a_k - a_{k+1}$ and in general

$$\Delta^{(n)} a_k = \Delta^{(n-1)} a_k - \Delta^{(n-1)} a_{k+1}.$$

One readily shows that

$$\Delta^{(n)} a_k = a_k - \binom{n}{1} a_{k+1} + \binom{n}{2} a_{k+2} - \ldots + (-1)^n \binom{n}{n} a_{k+n},$$

where

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Theorem 77. HYPOTHESIS: (1) $\sum_{n=0}^{\infty} (-1)^n a_n$ *converges to* A. CONCLUSION: $\sum_{n=0}^{\infty} \frac{\Delta^{(n)} a_0}{2^{n+1}}$ *converges to* A.

The $(-1)^n$ is inserted for convenience but is not intended to infer even the reality of the terms.

PROOF: Consider first $\frac{\Delta^{(n)} a_k}{2^n}$ and let $a_{n,p}$ of the lemma be $\frac{1}{2^n}\binom{n}{p}$ and $x_n = a_n$. We have the hypotheses of the lemma satisfied because

$$\frac{1}{2^n}\binom{n}{p} < \frac{1}{2^n} n^p \to 0$$

when $n \to \infty$ and,

$$\sum_{p=0}^{\infty} \frac{1}{2^n}\binom{n}{p} = \frac{1}{2^n}(1+1)^n = 1.$$

Of course $a_n \to 0$.

Hence,

$$\lim_{n\to\infty} \frac{1}{2^n}\Delta^{(n)} a_k = 0. \tag{2}$$

Next, set $a_{k,n} = (-1)^k\left[\frac{1}{2^n}\Delta^n a_k - \frac{1}{2^{n+1}}\Delta^{n+1} a_k\right]$.

Then, by (2),

$$\sum_{n=0}^{\infty} a_{k,n} = (-1)^k \frac{1}{2^0}\Delta^0 a_k = (-1)^k a_k.$$

We thus obtain an infinite series for each term of (1). We wish to apply Theorem 75 to the resulting array.

On account of the relation

$$\Delta^{(n+1)} a_k = \Delta^{(n)} a_k - \Delta^{(n)} a_{k+1},$$

we can write

$$\sum_{k=0}^{\infty} (-1)^k\left[\frac{1}{2^n}\Delta^{(n)} a_k - \frac{1}{2^{n+1}}\Delta^{(n+1)} a_k\right]$$

in the form

$$\sum_{k=0}^{\infty} \frac{1}{2^{n+1}}\left[(-1)^k \Delta^{(n)} a_k - (-1)^{k+1}\Delta^{(n)} a_{k+1}\right]$$

$$= \frac{1}{2^{n+1}}\left[\Delta^{(n)} a_0 - \lim_{k\to\infty}(-1)^k \Delta^{(n)} a_k\right].$$

Since $a_k \to 0$, $\Delta^{(n)} a_k \to 0$ when $k \to \infty$. Consequently,

$$\sum_{k=0}^{\infty} a_{k,n} = \frac{\Delta^{(n)}(a_0)}{2^{n+1}}.$$

In order for the theorem to be complete, it now is only necessary to show that R_P of Theorem 75 approaches zero. Using the notation of that theorem,

$$_P r_k = (-1)^k \frac{\Delta^{(P)} a_k}{2^{P+1}};$$

and consequently,

$$R_P = \frac{1}{2^{P+1}} \sum_{k=0}^{\infty} (-1)^k \Delta^{(P)} a_k.$$

Now, let $(-1)^k(a_k - a_{k+1} + a_{k+2} - \dots) = r_k$.

Write $\Delta^{(P)} a_k$ in expanded form and we have,

$$R_P = \frac{r_0 + \binom{P}{1} r_1 + \binom{P}{2} r_2 + \dots + \binom{P}{P} r_P}{2^{P+1}} \to 0$$

by the lemma since $r_P \to 0$.

§ 2. Multiplication of Series.

Theorem 78. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$ *both converge absolutely*; (ii) α_n *and* β_n *are two sequences so chosen that as* n *takes on the values* 1, 2, 3, ..., *every possible permutation that can be formed of two positive integers is presented once and only once by* α_n, β_n. CONCLUSION:

$$\sum_{n=1}^{\infty} a_{\alpha_n} b_{\beta_n}$$

converges absolutely to the value AB.

PROOF: Let $\sum_{n=1}^{\infty} |a_n| = \bar{A}$, $\sum_{n=1}^{\infty} |b_n| = \bar{B}$,

$$\sigma_m = \sum_{n=1}^{m} a_{\alpha_n} b_{\beta_n}, \quad s_n = \sum_{n=1}^{n} a_n, \quad t_n = \sum_{n=1}^{n} b_n,$$

and let any $\epsilon > 0$ be given. Choose m so large that whatever value n happens to have at the time, σ_m contains each term of the product

$$s_n t_n = (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n).$$

Then $\sigma_m = s_n t_n + R_{m,n}$, where $R_{m,n}$ is a sum of terms of the

form $a_{\alpha_j} b_{\beta_k}$, where at least one of α_j, β_k is greater than n. Then, with p chosen sufficiently large,

$$|R_{m,n}| \leqq [|a_{n+1}| + \ldots + |a_{n+p}|][|b_1| + \ldots + |b_{n+p}|] + [|b_{n+1}| + \ldots + |b_{n+p}|][|a_1| + \ldots + |a_{n+p}|].$$

Let δ be given and choose an M so that, when $n > M$, both

$$[|a_{n+1}| + \ldots + |a_{n+p}|] < \delta \text{ and } [|b_{n+1}| + \ldots + |b_{n+p}|] < \delta.$$

Moreover, $|a_1| + \ldots + |a_{n+p}| \leqq \bar{A}$

and $|b_1| + \ldots + |b_{n+p}| \leqq \bar{B}.$

Then $|\sigma_m - AB| \leqq |s_n t_n - AB| + |R_{m,n}| \leqq |s_n t_n - AB| + \delta(\bar{A} + \bar{B}).$

Now choose $N > M$ so that when $n > N$,

$$|s_n t_n - AB| < \frac{\epsilon}{2},$$

and let δ be so small that

$$\delta(\bar{A} + \bar{B}) < \frac{\epsilon}{2}.$$

Then $|\sigma_m - AB| < \epsilon,$

completing the proof.

This theorem can be loosely expressed by saying that absolutely convergent series can be multiplied together like polynomials.

In case one of the series is absolutely convergent and one conditionally convergent we have the following, but not so general, theorem.

Theorem 79. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n = A$ *converges absolutely*; (ii) $\sum_{n=1}^{\infty} b_n = B$ *converges*. CONCLUSION: $\sum_{n=1}^{\infty} c_n$ *converges to* AB, *where* $c_n = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1$.

PROOF: Corresponding to every n there are two positive numbers which we take as small as possible* and denote by ϵ_n and ϵ'_n, such that

$$|a_{n+q}| + \ldots + |a_{n+p}| \leqq \epsilon_n$$

and

$$|b_{n+q} + \ldots + b_{n+p}| \leqq \epsilon'_n$$

for every $p \geqq 0$ and $q \geqq 0$.

* In this connexion see the discussion of inferior limits in chapter XII.

Let $C_n = c_1 + \ldots + c_n$. Then

$$C_n = s_n t_n - a_2 b_n - a_3(b_{n-1} + b_n) - \ldots - a_n(b_2 + \ldots + b_n),$$

s_n and t_n having the same significance as in the previous theorem. From this

$$|C_n - AB| \leqq |s_n t_n - AB| + |a_2||b_n| + |a_3||b_{n-1} + b_n| + \ldots + |a_n||b_2 + b_3 + \ldots + b_n| \leqq |s_n t_n - AB| + |a_2|\epsilon'_n + |a_3|\epsilon'_{n-1} + \ldots + |a_n|\epsilon'_2.$$

According to Theorem 21 $\epsilon_n \to 0$ and $\epsilon'_n \to 0$. Let m be the largest integer contained in $\frac{n}{2}$, and let $G > \epsilon'_n$ for all values of n and let η_m be as small as possible but as large as any one of the set $\epsilon'_{n-m+2}, \epsilon'_{n-m+3}, \ldots \epsilon'_n$. Then

$$|C_n - AB| \leqq |s_n t_n - AB| + (|a_2| + \ldots + |a_m|)\eta_m + (|a_{m+1}| + \ldots + |a_n|)G \leqq |s_n t_n - AB| + \epsilon_2 \eta_m + \epsilon_{m+1} G.$$

When $n \to \infty$, $m \to \infty$. Consequently $\eta_m \to 0$ and $\epsilon_{m+1} \to 0$. Since $|s_n t_n - AB| \to 0$ also, $|C_n - AB| \to 0$ and the theorem is proved.

In case that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both are conditionally convergent and if $\sum_{n=1}^{\infty} c_n$ converges, it converges to the value AB. Before proceeding to the proof of this theorem we introduce three lemmas or subsidiary theorems. These theorems are of great interest in themselves, although their introduction at this point has primary reference to their immediate application.

Theorem 80. *Lemma* 1: HYPOTHESIS: $s_n \to s$. CONCLUSION:

$$S_n = \frac{s_1 + s_2 + \ldots + s_n}{n} \to s.$$

PROOF: Let $r_n = s - s_n$. Then

$$S_n = s - \frac{r_1 + r_2 + \ldots + r_n}{n}.$$

We consequently will have proved the theorem if we prove that

$$\frac{r_1 + r_2 + \ldots + r_n}{n} \to 0 \text{ if } r_n \to 0.$$

Let any $\epsilon > 0$ be given. By Theorem 6 there exists a $g > 0$ such that $g > |r_n|$, $n = 1, 2, 3, \ldots$, and if a $\delta > 0$ is given, an $M \geqq 1$ can be found such that when $n \geqq M$, $|r_n| \leqq \delta$. Choose a δ and let n be greater than the corresponding M.

$$|r_1 + \ldots + r_n| \leqq |r_1 + \ldots + r_{M-1}| + |r_M + \ldots + r_n|$$
$$\leqq (M-1)g + (n-M+1)\delta < Mg + n\delta.$$

Choose $\delta = \frac{\epsilon}{2}$ and let $n > \frac{2Mg}{\epsilon}$. Then

$$\left|\frac{r_1 + \ldots + r_n}{n}\right| < \epsilon,$$

which gives us the proof.

Theorem 81. *Lemma* 2: HYPOTHESIS: $A_n \to A$, $B_n \to B$. CONCLUSION:

(1)
$$\frac{A_1 B_n + \ldots + A_n B_1}{n} \to AB.$$

PROOF: $\frac{A_1 B_n + \ldots + A_n B_1}{n} - AB$

$$= \frac{(A_1 - A)(B_n - B) + \ldots + (A_n - A)(B_1 - B)}{n}$$
$$+ \frac{(A_1 + \ldots + A_n)B}{n} + \frac{(B_1 + \ldots + B_n)A}{n} - 2AB.$$

Consequently, by the previous theorem, this theorem will be established if it is shown that

$$\frac{(A_1 - A)(B_n - B) + \ldots + (A_n - A)(B_1 - B)}{n} \to 0.$$

This, however, is exactly the form (1), if $A_j - A$ is replaced by A_j and $B_j - B$ by B_j. We will do this and prove the following theorem.

HYPOTHESIS: $A_n \to 0$, $B_n \to 0$. CONCLUSION:

$$\frac{A_1 B_n + \ldots + A_n B_1}{n} \to 0.$$

Let $\epsilon > 0$ be given. There exists a $g > 0$ such that

$$|A_n| < g, \ |B_n| < g, \ n = 1, 2, 3, \ldots$$

and an M such that, when $n > M$, $|A_n| < \frac{\epsilon}{g}$ and $|B_n| < \frac{\epsilon}{g}$

simultaneously. Let $n > 2M + 1$. Then each term in $A_1 B_n + \ldots + A_n B_1$ will contain a factor with subscript greater than M. Hence

$$|A_1 B_n + \ldots + A_n B_1| < g \cdot \frac{\epsilon}{g} n = \epsilon n,$$

and hence
$$\left|\frac{A_1 B_n + \ldots + A_n B_1}{n}\right| < \epsilon,$$

which completes the proof, not only for the case that $A_n \to 0$, $B_n \to 0$, but also for the more general case that $A_n \to A$ and $B_n \to B$.

Theorem 82. *Lemma* 3. HYPOTHESES:

(i) $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$; (ii) $c_n = a_1 b_n + \ldots + a_n b_1$;

(iii) $C_n = c_1 + \ldots + c_n$. CONCLUSION: $\frac{C_1 + \ldots + C_n}{n} \to AB$.

PROOF: $C_\nu = \sum_{a=1}^{\nu} a_a (b_1 + \ldots + b_{\nu-a+1}) = \sum_{a=1}^{\nu} a_a B_{\nu-a+1}$

$$= \sum_{a=1}^{\nu} B_a a_{\nu-a+1}.$$

$$\frac{C_1 + \ldots + C_n}{n} = \frac{1}{n} \sum_{\nu=1}^{n} \sum_{a=1}^{\nu} B_a a_{\nu-a+1}$$
$$= \frac{1}{n} \sum_{a=1}^{n} \sum_{\nu=a}^{n} B_a a_{\nu-a+1} = \frac{1}{n} \sum_{a=1}^{n} B_a (a_1 + \ldots + a_{n-a+1})$$
$$= \frac{1}{n} \sum_{a=1}^{n} B_a A_{n-a+1} = \frac{B_1 A_n + \ldots + B_n A_1}{n} \to AB,$$

by Theorem 81.

We now are in a position to formally state and prove the theorem to which the three previous are lemmas.

Theorem 83. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$, *and* $\sum_{n=1}^{\infty} c_n$ *converges where* $c_n = a_1 b_n + \ldots + a_n b_1$. CONCLUSION: $\sum_{n=1}^{\infty} c_n = AB$.

PROOF: This theorem is an immediate consequence of lemma 1 and lemma 3.

The inverse process of multiplication is, of course, division. If we consider three series,

$$U = \sum_{n=1}^{\infty} u_n, \quad V = \sum_{n=1}^{\infty} v_n, \; v_1 \neq 0, \quad X = \sum_{n=1}^{\infty} x_n,$$

where $x_1 = \dfrac{u_1}{v_1}$ and when $n > 1$,

$$x_n = \frac{u_n - x_1 v_n - x_2 v_{n-1} - \dots - x_{n-1} v_2}{v_1},$$

and multiply formally by the rule that we have been using we obtain $U = XV$. The past theorem actually gives us the following.

Theorem 84. HYPOTHESIS: *Series U, V, and X converge.* CONCLUSION $U = XV$.

§ 3. Termwise multiplication of series.

Theorem 85. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} |b_n - b_{n+1}|$ *both converge.* CONCLUSION: $\sum_{n=1}^{\infty} a_n b_n$ *converges.*

PROOF: Let an $\epsilon > 0$ be given and let

$$s_n = \sum_{\nu=1}^{n} a_\nu.$$

Then $\qquad a_\nu = s_\nu - s_{\nu-1} = (s_\nu - s) - (s_{\nu-1} - s)$;

and hence $\displaystyle\sum_{\nu=n+1}^{n+p} a_\nu b_\nu = \sum_{\nu=n+1}^{n+p} [(s_\nu - s) - (s_{\nu-1} - s)]\, b_\nu$

$$= \sum_{\nu=n+1}^{n+p-1} (s_\nu - s)(b_\nu - b_{\nu+1}) - (s_n - s)\, b_{n+1} + (s_{n+p} - s)\, b_{n+p}.$$

Choose a $\delta > 0$; and then, let M be so chosen that when $n > M$, $|s_n - s| < \delta$. Moreover,

$$\sum_{\nu=n+1}^{n+p-1} |b_\nu - b_{\nu+1}| < B,$$

a certain fixed number, for any n and p. Consequently, when $n > M$ and $p \geqq 1$,

$$\left|\sum_{\nu=n+1}^{n+p} a_\nu b_\nu\right| < B\delta + \delta\,|b_{n+1}| + \delta\,|b_{n+p}|.$$

But $\sum_{n=1}^{\infty} |b_n - b_{n+1}|$ converges, and consequently

$$\sum_{n=1}^{\infty} (b_n - b_{n+1})$$

converges, that is $\sum_{n=1}^{m} (b_n - b_{n+1}) = b_1 - b_{m+1}$ approaches a limit as $m \to \infty$. This means that b_m approaches a limit and hence that there exists a c, such that $|b_m| < c$ always. Take this $c > B$; and then choose $\delta < \dfrac{\epsilon}{3c}$ and we have

$$\left|\sum_{\nu=n+1}^{n+p} a_\nu b_\nu\right| < \epsilon,$$

when $n > M$, thus completing the proof by Theorem 21.

Theorem 86. HYPOTHESES: (i) $\left|\sum_{n=0}^{n} a_n\right| < g$, *a fixed number*; (ii) $\sum_{n=0}^{\infty} |b_n - b_{n+1}| = G$ *converges*; (iii) $b_n \to 0$.

CONCLUSION: $\sum_{n=0}^{\infty} a_n b_n = S$ *converges and* $|S| \leqq gG$.

PROOF: Let $s_n = \sum_{n=0}^{n} a_n$. Then

$$\sum_{n=0}^{n-1} a_n b_n = b_0 s_0 + b_1 (s_1 - s_0) + b_2 (s_2 - s_1) + \dots$$

$$+ b_{n-1} (s_{n-1} - s_{n-2}) = \sum_{n=1}^{n-1} s_{n-1} (b_{n-1} - b_n) + b_{n-1} s_{n-1}.$$

Now $\qquad \displaystyle\sum_{n=1}^{n-1} |s_{n-1}|\,|b_{n-1} - b_n| \leqq gG.$

That is, $\qquad \displaystyle\sum_{n=1}^{\infty} s_{n-1} (b_{n-1} - b_n)$

converges absolutely and, by Theorem 26, the absolute value of its sum is less than or equal to gG. Since $b_{n-1}s_{n-1} \to 0$,

$$\sum_{n=0}^{\infty} a_n b_n$$

converges to the same value as

$$\sum_{n=1}^{\infty} s_{n-1}(b_{n-1}-b_n).$$

This completes the proof.

EXERCISES

153. Criticize the following:

$$\begin{aligned}\log 2 &= 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots\\ &= 1+(\tfrac{1}{2}-2(\tfrac{1}{2}))+\tfrac{1}{3}-(\tfrac{1}{4}-2(\tfrac{1}{4}))+\dots\\ &= [1+\tfrac{1}{2}+\tfrac{1}{3}+\dots]-2[\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\dots]\\ &= [1+\tfrac{1}{2}+\tfrac{1}{3}+\dots]-[1+\tfrac{1}{2}+\tfrac{1}{3}+\dots]=0.\end{aligned}$$

154. Give other examples illustrative of the errors in the work of the last exercise.

155–156. From your knowledge of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

what can you say of

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$$

and of $$\sum_{k=1}^{\infty}\left[\frac{1}{4k-3}-\frac{1}{4k-2}+\frac{1}{4k-1}-\frac{1}{4k}\right]?$$

157. Prove: Any conditionally convergent series can be converted into an absolutely convergent series by the proper insertion of brackets.

158–159. Using the transformation of Theorem 77, calculate to five decimal places

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots$$

and $$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots.$$

160. Prove: The sum of a conditionally convergent series of real terms is not altered if the displacement of the n-th term, multiplied by the absolute value of that subsequent term having the greatest absolute value, approaches zero when $n \to \infty$.

161. Prove: $\sum_{n=1}^{\infty} \frac{a_n+a_{n+1}}{2}$ converges if $\sum_{n=1}^{\infty} a_n$ converges.

162. Discuss the converse of the theorem of Exercise 161.

163. If $a_n \to a$ and $b_n \to b$ and

$$a_1b_1+a_2(b_2-b_1)+a_3(b_3-b_2)+\dots$$

converges to S, prove that

$$b_1(a_1-a_2)+b_2(a_2-a_3)+b_3(a_3-a_4)+\dots$$

converges. Determine the sum to which it converges.

164. Prove: If $\sum_{n=1}^{\infty} a_n$ converges and if $a_n > a_{n+1}$,

$$\sum_{n=1}^{\infty} n(a_n-a_{n+1})$$

converges.

165, 166. Arrange the terms of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ so that the new series will have the sum 10. Arrange the terms of $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$ so that the new series will have the sum 0.

167. Show by an example that $\sum_{k=0}^{\infty} \frac{\Delta^k a_0}{2^{k+1}}$ may converge when $\sum_{n=0}^{\infty} (-1)^n a_n$ diverges.

168. Discuss the Cauchy product (Theorem 83) formed for the two series

$$1-\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n \text{ and } 1+\sum_{n=1}^{\infty}\left(2^n+\frac{1}{2^{n+1}}\right)\left(\frac{3}{2}\right)^{n-1}.$$

169–172. Using Maclauren's formula from calculus, having found series expansions for e^x, $\cos x$, $\sin x$, $\log(1+x)$, $\sin^{-1} x$, by multiplication find series expansions for $e^x \cos x$, $e^x \sin x$, $(1+x)\log(1+x)$, $\cos x \sin^{-1} x$. Draw what conclusions you can relative to convergence.

173. Prove:

$$[1+x+x^2+x^3+\dots]^2 = 1+2x+3x^2+4x^3+\dots$$

when $|x| < 1$.

174. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\beta}$ can be multiplied by the Cauchy rule (Theorem 83) when and only when $\alpha+\beta > 1$.

175. Show that

$$\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots)^2$$
$$= \tfrac{1}{2}-\tfrac{1}{3}(1+\tfrac{1}{2})+\tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})-\tfrac{1}{5}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\dots,$$

formally.

Test this series for convergence and draw conclusions relative to the use of the equality mark.

176–177. Considering the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ and } \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

$a_n > a_{n+1} \to 0$, $b_n > b_{n+1} \to 0$; prove that for the convergence of the product series formed according to Theorem 83: (1) it is necessary and sufficient that

$$w_n = a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n \to 0:$$

(2) it is necessary and sufficient that both

$$a_n(b_1+\dots+b_n) \to 0 \text{ and } b_n(a_1+\dots+a_n) \to 0.$$

178. Is the following theorem true or false: If $b_n \to 0$ but $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{r=0}^{\infty} (a_1 b_{kr+1} + a_2 b_{kr+2} + \dots + a_k b_{kr+k})$ converges, if and only if $a_1+a_2+\dots+a_k = 0$.

CHAPTER VIII

MULTIPLE SERIES

§ 1. Double sequences and series.

Definition 14. *If each permutation (p, q) of any two positive integers determines a particular number, we say that we have a double, or two-dimensional, infinite sequence.*

We can denote the sequence simply by $a_{p,q}$. Clearly it is not necessary that p and q be positive integers. Any numbers that are put into one to one correspondence with the positive integers will do.

Definition 15. *The double sequence $a_{p,q}$ is said to approach the limit A, if corresponding to any $\epsilon > 0$ there exist two numbers P and Q, such that, when $p > P$ and $q > Q$ simultaneously, $|a_{p,q} - A| < \epsilon$. We write $\lim_{p\to\infty, q\to\infty} a_{p,q} = A$.*

The following three theorems are proved similarly to corresponding theorems for simple sequences. We shall number them as one theorem.

Theorem 87. *A necessary and sufficient condition that a double sequence $a_{p,q}$ converge, is that, given any $\epsilon > 0$, it is possible to find (1) a P and a Q, such that, whenever $p > P$, $q > Q$, $r > 0$, and $s > 0$, $|a_{p+r,q+s} - a_{p,q}| < \epsilon$; or (2) a number M, such that, when $p > M$, $r > 0$, and $s > 0$,*

$$|a_{p+r,\,p+s} - a_{p,p}| < \epsilon;$$

or (3) a number m, such that, when $p > m$ and $q > m$,

$$|a_{p,q} - a_{m,m}| < \epsilon.$$

Proof is omitted.

It is to be remarked, that

$$\lim_{p\to\infty,\,q\to\infty} a_{p,q}, \quad \lim_{p\to\infty}(\lim_{q\to\infty} a_{p,q}), \text{ and } \lim_{q\to\infty}(\lim_{p\to\infty} a_{p,q})$$

have different meanings. The reader should set up illustrations displaying this difference.

Definition 16. *The totality of numbers $a_{p,q}$ constitute an infinite double series or simply a double series.*

It can be displayed in the following manner:

$$(1)\qquad \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Each $a_{p,q}$ is called a term of the double series. We shall speak of the rows and columns of a double series. This terminology applies to the array (1) and has an evident significance.

Definition 17. *The double series* (1) *is said to converge if all simple series chosen from the terms of* (1), *so as to include every term of* (1) *whatsoever, have the same sum. This number is called the sum of* (1). *The series is said to converge to its sum. If* (1) *does not converge it is said to diverge.*

Examples of such simple series are

$$(2)\qquad a_{11}+a_{12}+a_{22}+a_{21}+a_{13}+a_{23}+a_{33}+a_{32}+a_{31}+a_{14}+a_{24} \\ +a_{34}+a_{44}+a_{43}+a_{42}+a_{41}+\cdots.$$

$$(3)\qquad a_{11}+a_{12}+a_{21}+a_{13}+a_{22}+a_{31}+a_{14}+a_{23}+a_{32}+a_{41}+\cdots.$$

In deference to convention the double series (1) will be written with plus signs thus:

$$(4)\qquad \begin{array}{l} a_{11}+a_{12}+a_{13}+\cdots \\ +a_{21}+a_{22}+a_{23}+\cdots \\ +a_{31}+a_{32}+a_{33}+\cdots \\ + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

It is to be remarked, however, that the plus signs play no greater part here than in the case of simple series, as explained in section 1 of Chapter III. A second and more compact notation is $\sum\limits_{i=1,j=1}^{\infty} a_{ij}$. Neither notation is to imply convergence.

Changing rows to columns and columns to rows in no way affects a double series, it simply changes the form in which the terms of the series are displayed. Any theorem proved with reference to the rows and columns of a double series can be restated changing the word row to column and notation for row to that for column, and conversely.

Theorem 88. *A necessary and sufficient condition that the double series* (1) *converge, is that the double series whose terms are the absolute values of the terms of* (1) *converge; that is, that the series*

$$(5)\qquad \begin{array}{l} |a_{11}|+|a_{12}|+|a_{13}|+\cdots \\ +|a_{21}|+|a_{22}|+|a_{23}|+\cdots \\ +|a_{31}|+|a_{32}|+|a_{33}|+\cdots \\ + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

converge.

PROOF: (i) Necessary:

Let $\sum\limits_{n=1}^{\infty} b_n = B$ be a simple series whose terms are chosen from the terms of (1), omitting none, as explained in Definition 17. This necessarily converges absolutely, otherwise we could rearrange its terms, having $\sum\limits_{n=1}^{\infty} b_{\lambda_n} \neq B$ by Theorem 69,

Corollary 3. Since $\sum\limits_{n=1}^{\infty} |b_n|$ converges, every simple series chosen from the terms of (5), omitting none, converges to B; and hence, by Definition 17, (5) converges.

(ii) Sufficient:

This is an immediate consequence of Theorem 67 and Definition 17.

As a result of the theorem just proved the examination of a double series for convergence or divergence generally begins with an examination of the series of absolute values. This problem can be reduced by the definition of convergence to the examination of simple series with positive terms. There are, however, some other useful theorems which we proceed to develop.

Theorem 89. *Let*

$$
\begin{aligned}
(6) \qquad & a_{11}+a_{12}+a_{13}+\dots \\
& +a_{21}+a_{22}+a_{23}+\dots \\
& +a_{31}+a_{32}+a_{33}+\dots \\
& + \quad . \quad . \quad . \quad .
\end{aligned}
$$

be a double series. HYPOTHESES: (i) *All the terms are positive,* $a_{ij}>0$; (ii) $a_{i1}+a_{i2}+\dots=A_i$ *converges*; (iii) $A_1+A_2+\dots=A$ *converges.* CONCLUSION: (6) *converges.*

PROOF: Form the simple series

$$a_{11}+a_{12}+a_{21}+a_{13}+a_{22}+a_{31}+\dots;$$

that is, follow the diagonals in the rectangular array from the upper right hand to the lower left hand. Denote the sum of the first n terms of this series by s_n, then $s_n \leqq A$ always and hence approaches a limit. Any other series would simply be a rearrangement of the terms of this one and would converge to the same value. The double series consequently converges by Definition 17.

Theorem 90. *Let*

$$
\begin{aligned}
(7) \qquad & a_{11}+a_{12}+a_{13}+\dots \\
& +a_{21}+a_{22}+a_{23}+\dots \\
& +a_{31}+a_{32}+a_{33}+\dots \\
& + \quad . \quad . \quad .
\end{aligned}
$$

be a double series converging to A. No hypothesis as to the nature of the terms is made. CONCLUSIONS:

(i) $a_{i1}+a_{i2}+a_{i3}+\dots,\ i=1,2,3,\dots,$

converges absolutely to a value which we denote by A_i; (ii) $A_1+A_2+A_3+\dots$ *converges to* A.

PROOF: (i) Suppose the double series of the absolute values of the terms of (7) converges to B, then

$$|a_{i1}|+|a_{i2}|+\dots+|a_{in}| \leqq B$$

for any i. Consequently, $\sum_{n=1}^{\infty}|a_{in}|$ converges, by Theorem 17; and consequently, $\sum_{n=1}^{\infty} a_{in}$ converges, by Theorem 26.

(ii) Let p be any fixed positive integer. Then

$$
\begin{aligned}
& |a_{11}|+\dots+|a_{1n}| \\
& + \quad . \quad . \quad . \quad . \\
& +|a_{p1}|+\dots+|a_{pn}| \leqq B.
\end{aligned}
$$

Consequently, by Theorem 8,

$$|A_1|+\dots+|A_p| \leqq B.$$

Hence, by Theorems 17 and 26, $A_1+A_2+\dots$ converges. Suppose that the value to which it converges is $\bar{A}$.

To prove $\bar{A}=A$: Let $\epsilon>0$ be given. Let

$$\sigma_n = A_1+\dots+A_n$$

and choose an m so that, when $n \geqq m$,

$$(8) \qquad |\sigma_n-\bar{A}| < \frac{\epsilon}{3}.$$

Next consider the simple series formed by following successively the sides of rectangles in the two-dimensional array beginning in the upper left-hand corner. This series converges absolutely, and hence we can take so many terms in it that it is impossible to get a sum out of the terms remaining in the double series in absolute value as great as $\frac{\epsilon}{3}$. In other words, one can mark off a square in the upper left-hand corner of the array so that it is impossible to get a sum of terms outside this square in absolute value as large as $\frac{\epsilon}{3}$. Suppose such a square to have P elements on a side, and let $P>m$. Denote the sum of the elements in any rectangle in the upper left-hand corner of the array of p rows and q columns by ${}_pS_q$. Let p and q both be greater than P. Then,

$$(9) \qquad |A-{}_pS_q| \leqq \frac{\epsilon}{3}.$$

Let $A_i-(a_{i1}+\dots+a_{in})=r_n^{(i)}$. Hold p fast and take Q so large that when $i \leqq p$ and $q>Q$, $|r_q^{(i)}|<\frac{\epsilon}{3p}$. Then,

$$(10) \qquad |{}_pS_q-\sigma_p| < \frac{\epsilon}{3}.$$

Then, from (8), (9), and (10),

$$|A-\bar{A}| \leqq |\sigma_p - \bar{A}| + |{}_pS_q - \sigma_p| + |A - {}_pS_q| < \epsilon.$$

Hence $$\bar{A} = A.$$

In connexion with this theorem it would be well to turn back to Theorem 75.

We shall next state without proof three theorems which are more or less easy generalizations of theorems proved for simple series.

Theorem 91. HYPOTHESES: (i) $\sum_{n=1,\, m=1}^{\infty} a_{m,n}$ *converges to* A; (ii) $a_{m,n} > 0$ *for all values of* m *and* n; (iii) $|b_{m,n}| \leqq a_{m,n}$ *when* $n > k$, $m > l$. CONCLUSION: (i) $\sum_{n=1,\, m=1}^{\infty} b_{m,n}$ *converges.*

Theorem 92. HYPOTHESES: (i) $\sum_{n=1,\, m=1}^{\infty} a_{m,n}$ *diverges.* (ii) $|b_{m,n}| \geqq |a_{m,n}|$ *when* $n > k$, $m > l$. CONCLUSION:

$$\sum_{n=1,\, m=1}^{\infty} b_{m,n} \text{ diverges.}$$

Theorem 93. *Let* $f(x, y) \geqq f(x', y') \geqq 0$ *when* $x \geqq x' \geqq a$ *and* $y \geqq y' \geqq b$. *Let* S *be a region in the xy-plane, with Y-axis positive downward, bounded by the lines* $x = a$, $y = b$, *and a simple curve c, so drawn as to cut the line* $y = b$ *to the right of the point* (a, b) *and the line* $x = a$ *below* (a, b) *and neither line elsewhere. Then, a sufficient condition that the double series* $\sum_{j=a+1,\, k=b+1}^{\infty} f(j, k)$ *converge, is that* $\int_S\int f(x, y)\, dA$ *approach a limit as c recedes from the origin in any way so that the distance of each point on it from the origin becomes infinite. A sufficient condition that* $\sum_{j=a,\, k=b}^{\infty} f(j, k)$ *diverge, is that* $\int_S\int f(x, y)\, dA$ *become infinite as c recedes.*

The following two examples illustrate vividly the necessity for great care in handling double series.

(11)

$$\left(-\frac{1}{2}\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{3}\cdot\frac{2}{3}\right) + \left(\frac{1}{3}\cdot\frac{2}{3} - \frac{1}{4}\cdot\frac{3}{4}\right) + \ldots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{2^2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2^2} - \frac{1}{3}\cdot\left(\frac{2}{3}\right)^2\right) + \left(\frac{1}{3}\cdot\left(\frac{2}{3}\right)^2 - \frac{1}{4}\left(\frac{3}{4}\right)^2\right) + \ldots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{2^3}\right) + \left(\frac{1}{2}\cdot\frac{1}{2^3} - \frac{1}{3}\cdot\left(\frac{2}{3}\right)^3\right) + \left(\frac{1}{3}\left(\frac{2}{3}\right)^3 - \frac{1}{4}\left(\frac{3}{4}\right)^3\right) + \ldots$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Consider the simple series formed of the rows. Denoting by $s_n^{(i)}$ the sum of the first n terms in the i-th row, we have

$$s_n^{(i)} = -\frac{1}{n+1}\left(\frac{n}{n+1}\right)^i \to 0.$$

The series of values, the $A_1 + A_2 + \ldots$ of the previous theorem, is $0 + 0 + 0 + \ldots = 0$. Now, however, consider the simple series consisting of the columns. Each column consists of two geometric series. The sum of the i-th column is

$$\frac{i-1}{i} - \frac{i}{i+1}.$$

The sum of the series of these values, namely

$$-\tfrac{1}{2} + (\tfrac{1}{2} - \tfrac{2}{3}) + (\tfrac{2}{3} - \tfrac{3}{4}) + \ldots,$$

is -1. This differs from the sum previously obtained when we summed by rows. The double series must diverge; otherwise, by Theorem 90, these two processes would have yielded the same result.

Consider next,

$$\begin{array}{ll} & 1 - \tfrac{1}{2} + 0 + 0 + \ldots \\ (12) & + 0 + \tfrac{1}{3} - \tfrac{1}{4} + 0 + \ldots \\ & + 0 + 0 + \tfrac{1}{5} - \tfrac{1}{6} + \ldots \\ & + \; . \quad . \quad . \quad . \end{array}$$

Here each row and each column, considered as a simple series, converges absolutely and the series of values in each instance converges absolutely and to the same value. The double series, however, diverges.

Double series have many applications in the study of simple series. For example, consider the following proof of Theorem 78 relative to the multiplication of two absolutely convergent series. Let the series be $a_1 + a_2 + a_3 + \ldots = A$ and $b_1 + b_2 + b_3 + \ldots = B$. Write down the double series

$$
(13) \qquad \begin{array}{l}
\, b_1 a_1 + b_1 a_2 + b_1 a_3 + \ldots \\
+ b_2 a_1 + b_2 a_2 + b_2 a_3 + \ldots \\
+ b_3 a_1 + b_3 a_2 + b_3 a_3 + \ldots \\
+ \; . \quad . \quad . \quad . \quad . \quad .
\end{array}
$$

This double series converges; for let

$$\sum_{n=1}^{\infty} |a_n| = \bar{A} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| = \bar{B}$$

and consider

$$
(14) \qquad \begin{array}{l}
\, |b_1||a_1| + |b_1||a_2| + |b_1||a_3| + \cdots \\
+ |b_2||a_1| + |b_2||a_2| + |b_2||a_3| + \cdots \\
+ |b_3||a_1| + |b_3||a_2| + |b_3||a_3| + \cdots \\
+ \; . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}
$$

Sum by rows according to Theorem 90. We get the value $\bar{A}\bar{B}$ which proves convergence not only of (14) but of (13) also. Consequently (13) can be summed by rows, by Theorem 90. The value obtained is AB. The simple series formed by following successively the diagonals from upper right hand to lower left hand, as indicated by arrows, must converge to the value AB also, but this is a series such as is considered in Theorem 78.

§ 2. Higher-dimensional series.

Definition 18. *If corresponding to each permutation of any three positive integers i, j, k there is a number a_{ijk}, the three-dimensional array* $\displaystyle\sum_{i=1,\, j=1,\, k=1}^{\infty} a_{ijk}$, *where i, j, k range independently over all positive integral values, is called a triple series. The a_{ijk}'s are called the terms of the series.*

We shall think of these terms as arranged in space so that the term a_{ijk} occupies the position whose Cartesian coordinates are (i, j, k). We shall speak of a plane meaning those elements which in this geometric arrangement lie in a plane parallel to one of the coordinate planes, and of a column as those elements lying in a line parallel to one of the coordinate axes.

A generalization to series of higher dimensions is easy. The whole theory is, for that matter, a generalization of double series involving little that is additional. The theorems are a result of the definition of convergence which is the same for series of two and higher dimensions.

Definition 19. *An n-dimensional series is said to converge if every simple series formed of its terms, so as to include each of them once and only once as a term, converges to the same value. This number is called its sum.*

When an n-dimensional series does not converge it is said to diverge.

The following are three fundamental theorems. For simplicity we shall state them for triple series. Proofs are omitted, as they follow so closely those given for the corresponding theorems on double series.

Theorem 94. *A necessary and sufficient condition that a triple series converge is, that it converge absolutely; that is, that the corresponding triple series of absolute values converge.*

Theorem 95. HYPOTHESIS: $a_{i,j,k} > 0$ *for all values of i, j, k.* CONCLUSION: *A necessary and sufficient condition for* $\displaystyle\sum_{i=1,\, j=1,\, k=1}^{\infty} a_{ijk}$ *to converge is, that* $\displaystyle\sum_{i=1}^{\infty}\Big(\sum_{j=1,\, k=1}^{\infty} a_{ijk}\Big)$ *or* $\displaystyle\sum_{i=1,\, j=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{ijk}\Big)$ *or other similar series formed by permuting i, j, k, converge.*

Theorem 96. HYPOTHESIS: $\sum_{i=1, j=1, k=1}^{\infty} a_{ijk}$, *where on assumption is made relative to the a_{ijk}'s, converges to A.*

CONCLUSION:

$$A = \sum_{i=1}^{\infty}\left(\sum_{j=1, k=1}^{\infty} a_{ijk}\right) = \sum_{i=1, j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{ijk}\right)$$

and similar series where i, j, k are permuted.

$$\sum_{i=1}^{\infty}\left(\sum_{j=1, k=1}^{\infty} a_{ijk}\right)$$

can be expressed by saying that we sum by planes parallel to the J-K-plane, and $\sum_{i=1, j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{ijk}\right)$ by saying that we sum by columns parallel to the K-axis.

EXERCISES

Consider the convergence or divergence of the following double series; wherever possible, sum by rows and also by columns.

179.
$$\begin{array}{l} 1 - \frac{1}{2^2} + 0 + 0 + \frac{1}{2^5} - \frac{1}{2^6} + 0 + 0 + \frac{1}{2^9} - \frac{1}{2^{10}} + \dots \\ -\frac{1}{2^2} + \frac{1}{2^3} + 0 + 0 - \frac{1}{2^6} + \frac{1}{2^7} + 0 + 0 - \frac{1}{2^{10}} + \frac{1}{2^{11}} + \dots \\ +\frac{1}{2^3} - \frac{1}{2^4} + 0 + 0 + \frac{1}{2^7} - \frac{1}{2^8} + 0 + 0 + \frac{1}{2^{11}} - \frac{1}{2^{12}} + \dots \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}$$

180.
$$\begin{array}{l} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \\ +0+0+0+0+0+\dots \\ -\frac{1}{2!} - \frac{1}{2!} - \frac{1}{(2!)(2!)} - \frac{1}{(2!)(3!)} - \frac{1}{(2!)(4!)} - \dots \\ +0+0+0+0+0+\dots \\ +\frac{1}{4!} + \frac{1}{4!} + \frac{1}{(4!)(2!)} + \frac{1}{(4!)(3!)} + \frac{1}{(4!)(4!)} + \dots \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}$$

181.
$$\begin{array}{l} 1+2+4+8+\dots \\ -\frac{1}{2}-1-2-4-\dots \\ +\frac{1}{4}+\frac{1}{2}+1+2+\dots \\ -\frac{1}{8}-\frac{1}{4}-\frac{1}{2}-1-\dots \\ + \quad . \quad . \quad . \quad . \end{array}$$

182.
$$\begin{array}{l} 0+1+0+0+0+\dots \\ -1+0+1+0+0+\dots \\ +0-1+0+1+0+\dots \\ +0+0-1+0+1+\dots \\ + \quad . \quad . \quad . \quad . \end{array}$$

183.
$$\begin{array}{l} -2+1+0+0+0+\dots \\ +1-2+1+0+0+\dots \\ +0+1-2+1+0+\dots \\ +0+0+1-2+1+\dots \\ +0+0+0+1-2+\dots \\ + \quad . \quad . \quad . \quad . \end{array}$$

184.
$$\begin{array}{l} 2+0-1+0+0+0+\dots \\ +0+2+0-1+0+0+\dots \\ -1+0+2+0-1+0+\dots \\ +0-1+0+2+0-1+\dots \\ +0+0-1+0+2+0-\dots \\ + \quad . \quad . \quad . \quad . \quad . \end{array}$$

185.
$$\begin{array}{l} \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots \\ +0+\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots \\ +0+0+\frac{1}{3.4} + \frac{1}{4.5} + \dots \\ +0+0+0+\frac{1}{4.5} + \dots \\ + \quad . \quad . \quad . \quad . \quad . \end{array}$$

186.
$$\begin{array}{l} 1+1+1+1+\dots \\ +1-1-1-1-\dots \\ +1-1+0+0+\dots \\ +1-1+0+0+\dots \\ +\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

185.
$$\begin{array}{l} \frac{1}{2}-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}+\frac{1}{8}-\frac{1}{16}+\frac{1}{16}-\dots \\ +\frac{1}{2^2}-\frac{3}{4^2}+\frac{3}{4^2}-\frac{7}{8^2}+\frac{7}{8^2}-\frac{15}{16^2}+\frac{15}{16^2}-\dots \\ +\frac{1}{2^3}-\frac{3^2}{4^3}+\frac{3^2}{4^3}-\frac{7^2}{8^3}+\frac{7^2}{8^3}-\frac{15^2}{16^3}+\frac{15^2}{16^3}-\dots \\ +\frac{1}{2^4}-\frac{3^3}{4^4}+\frac{3^3}{4^4}-\frac{7^3}{8^4}+\frac{7^3}{8^4}-\frac{15^3}{16^4}+\frac{15^3}{16^4}-\dots \\ +\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

188.
$$\begin{array}{l} 0+\frac{1}{1^2-2^2}+\frac{1}{1^2-3^2}+\frac{1}{1^2-4^2}+\dots \\ +\frac{1}{2^2-1^2}+0+\frac{1}{2^2-3^2}+\frac{1}{2^2-4^2}+\dots \\ +\frac{1}{3^2-1^2}+\frac{1}{3^2-2^2}+0+\frac{1}{3^2-4^2}+\dots \\ +\frac{1}{4^2-1^2}+\frac{1}{4^2-2^2}+\frac{1}{4^2-3^2}+0+\dots \\ +\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

189. Substitute:

(i) $y=b_0+b_1x+b_2x^2+\dots$

in (ii) $a_0+a_1y+a_2y^2+a_3y^3+\dots,$

obtaining formally

$$\begin{array}{l} a_0 + 0 + 0 + \dots \\ +a_1b_0+a_1b_1x + a_1b_2x^2 + \dots \\ +a_2b_0^2+2a_2b_0b_1x+a_2(b_1^2+2b_0b_2)x^2 + \dots \\ +a_3b_0^3+3a_3b_0^2b_1x+3a_3(b_0b_1^2+b_0^2b_2)x^2+\dots \\ +\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

Discuss the convergence of this double series for particular series (1) and (2) of your own choosing.

190. The following definition is sometimes given for the convergence of a double series. Given

$$\begin{array}{l} a_{11}+a_{12}+a_{13}+\dots \\ +a_{21}+a_{22}+a_{23}+\dots \\ +a_{31}+a_{32}+a_{33}+\dots \\ +\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

Mark off a rectangle of m rows and n columns in the upper left-hand corner of this array. Denote the sum of all the terms in this rectangle by $s_{m,n}$. If $s_{m,n}$ approaches a limit, s, according to Definition 15 the double series is said to converge and to have the sum s. It is said to diverge in the contrary case.

This definition is frequently called the Pringsheim definition in distinction from the definition of the text, which is called the Jordan definition.

Are the Pringsheim and Jordan definitions equivalent? Illustrate.

191. If each term of the double series is positive, are the Pringsheim and Jordan definitions equivalent?

192. Test series of Exercises 179 to 188 according to the Pringsheim definition.

193. By means of the theory of double series, prove Theorem 79.

CHAPTER IX

UNIFORM CONVERGENCE

§ 1. Definitions and necessary and sufficient conditions.

Let $s_n(z)$ be an infinite sequence defined for a set of values of z, which set we call I. Let $s_n(z) \to f(z)$, and let

$$r_n(z) = f(z) - s_n(z).$$

Definition 20. *$s_n(z)$ is said to approach $f(z)$ uniformly over I if, when any particular $\epsilon > 0$ is given, there exists a positive number M, such that when $n > M$, $|r_n(z)| < \epsilon$ for all values of z in the set I simultaneously.*

Now let $u_1(z)$, $u_2(z)$, ... be functions defined for all values of z in I, and let $s_n(z) = u_1(z) + \ldots + u_n(z)$. Let $\sum_{n=1}^{\infty} u_n(z)$ converge for all values of z in I. Let

$$\sum_{n=1}^{\infty} u_n(z) = f(z) \text{ and } r_n(z) = f(z) - s_n(z).$$

Definition 21.

$$\sum_{n=1}^{\infty} u_n(z) \tag{1}$$

converges uniformly over I, if $s_n(z)$ approaches $f(z)$ uniformly over I.

Here, as generally, the sequence is fundamental. However, we shall immediately throw the discussion over to series and make our study a study primarily of series. An interpretation in terms of the sequence can readily be given at all times.

Either of the following two theorems might be made the basis of a satisfactory definition, and the given definition made a necessary and sufficient condition.

Theorem 96. *A necessary and sufficient condition that the series* (1) *be uniformly convergent over I is, that when any $\epsilon > 0$ is given, it is possible to find an M, such that whenever n and n' are both greater than M,*

$$|s_n(z) - s_{n'}(z)| < \epsilon$$

for every value of z in I simultaneously.

PROOF: I. The condition is necessary.

Choose M so large that when $n > M$,

$$|r_n(z)| < \frac{\epsilon}{2}$$

for every value of z in I simultaneously. Then since

$$r_n(z) = f(z) - s_n(z), \quad |f(z) - s_n(z)| < \frac{\epsilon}{2}, \text{ and}$$

$$|f(z) - s_{n'}(z)| < \frac{\epsilon}{2},$$

when $n, n' > M$. Combining, $|s_n(z) - s_{n'}(z)| < \epsilon$.

II. The condition is sufficient.

Let $\bar{\epsilon}$ be given and choose M such that when n and n' are both greater than M,

$$|s_n(z) - s_{n'}(z)| < \bar{\epsilon} < \epsilon$$

for every value of z in I simultaneously. Let $n' \to \infty$, then

$$|s_n(z) - f(z)| = |r_n(z)| \leqq \bar{\epsilon} < \epsilon.$$

Theorem 97. *A necessary and sufficient condition that* (1) *be uniformly convergent over I is, that when any $\epsilon > 0$ is given, it is possible to find an M, such that when $n > M$,*

$$|s_n(z) - s_M(z)| < \epsilon$$

simultaneously for every z in I.

PROOF: I. The condition is necessary.

Under I of the previous theorem simply replace n' by M.

II. The condition is sufficient.

Choose M so that when $n > M$, $|s_n(z) - s_M(z)| < \frac{\epsilon}{2}$ for all values of z in I simultaneously. Then let n and n' both be greater than M. Then

$$|s_n(z) - s_M(z)| < \frac{\epsilon}{2} \text{ and } |s_{n'}(z) - s_M(z)| < \frac{\epsilon}{2}.$$

Combining, $\qquad |s_n(z) - s_{n'}(z)| < \epsilon.$

Proof follows from the previous theorem.

An example of a non-uniformly convergent series is the following. Further examples will appear later. Consider

$$x^2(1-x^2)+x^2(1-x^2)^2+x^2(1-x^2)^3+\dots,$$

where x is real.

It is a geometric series with ratio $(1-x^2)$. It converges when $|x|<\sqrt{2}$. Denote its sum by $f(x)$. Then $f(0)=0$, and when $x\neq 0$, $f(x)=1-x^2$. It is to be noticed that $f(x)$ is discontinuous at $x=0$, despite the fact that each term of the series is continuous.

To show that the series is not uniformly convergent we adopt the method of indirect proof. Assume the series uniformly convergent over the interval defined by the inequalities

$$(2) \qquad -1\leqq x\leqq 1.$$

Denote, as usual, the remainder after n terms by $r_n(x)$, and let an $\epsilon>0$ but $<\frac{1}{2}$ be given. Then choose M, so that when $n>M$, $|r_n(x)|<\epsilon$ for all values of x in (2) simultaneously. But, when $x\neq 0$,

$$|r_n(x)|=|(1-x^2)|^{n+1}=(1-x^2)^{n+1},$$

and when $\quad 0<x<\sqrt{1-\frac{1}{\sqrt[n+1]{2}}},\quad (1-x^2)^{n+1}>\frac{1}{2}$

for any n, that is $|r_n(x)|>\frac{1}{2}$. But $|r_n(x)|<\frac{1}{2}$ when $n>M$. As a result of this contradiction the series is seen not to be uniformly convergent over (2).

§ 2. Simpler theorems.

Theorem 98. HYPOTHESES: (i) $\sum_{n=1}^{\infty}u_n(z)$ *converges at the points of a set I*; (ii) *The points of I are finite in number.* CONCLUSION: $\sum_{n=1}^{\infty}u_n(z)$ *converges uniformly over I.*

PROOF: Denote the points of I by $z_1, z_2, \dots, z_m$. Let $\epsilon>0$ be given and let M_i, $i=1, 2, \dots, m$, be so chosen that when $n>M_i$, $|r_n(z_i)|<\epsilon$. Let $M\geqq M_i$, $i=1, \dots, m$. Then, when $n>M$, $|r_n(z)|<\epsilon$ simultaneously for all values of z in I and proof is complete.

Theorem 99. HYPOTHESIS: $\sum_{n=1}^{\infty}u_n(z)$ *converges uniformly over each of sets $I_1, I_2, \dots, I_m$.* CONCLUSION: *It converges uniformly over the set I composed of $I_1, I_2, \dots, I_m$ taken together.*

Proof is similar to that of the previous theorem and is omitted.

Theorem 100. HYPOTHESIS: $\sum_{n=1}^{\infty}u_n^{(i)}(z)$, $i=1, 2, \dots, k$ *converge uniformly over a set I.* CONCLUSION:

$$(3) \qquad \sum_{n=1}^{\infty}\left(u_n^{(1)}(z)+u_n^{(2)}(z)+\dots+u_n^{(k)}(z)\right)$$

converges uniformly over I.

PROOF: Let $\epsilon>0$ be given and let

$$s_n^{(i)}(z)=\sum_{n=1}^{n}u_n^{(i)}(z).$$

Choose M so large that $|s_n^{(i)}(z)-s_{n'}^{(i)}(z)|<\frac{\epsilon}{k}$ for all z's of I and for $i=1, 2, \dots, k$ simultaneously when $n, n'>M$. To do this choose M separately for each series and then take the largest. Let s_n be the sum of the first n terms of (3). Then, when $n, n'>M$,

$$|s_n(z)-s_{n'}(z)|\leqq\sum_{i=1}^{k}|s_n^{(i)}(z)-s_{n'}^{(i)}(z)|<\epsilon,$$

establishing the theorem.

The following theorems, numbers 101 to 110, can easily be established. Due to their simplicity proofs are omitted. Some of these theorems are stated for the series and some for the sequence.

Theorem 101. *Using the notation of the last theorem.* HYPOTHESES: (i) $\sum_{n=1}^{\infty}\left(u_n^{(1)}(z)+\dots+u_n^{(k)}(z)\right)$ *converges uniformly over I*; (ii) $u_n^{(i)}(z)\geqq 0$, $i=1, \dots, k$, $n=1, 2, \dots$, *and z of I.* CONCLUSION: $\sum_{n=1}^{\infty}u_n^{(i)}(z)$ *converges uniformly over I.*

In both this theorem and the next k may become infinite when n becomes infinite.

Theorem 102. Retaining the same notation. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n^{(j)}(z)$ *converges non-uniformly over* I, *where* j *is any one of the numbers* 1, 2, ..., k; (ii) $u_n^{(j)}(z) \geqq 0$, $j = 1, 2, \ldots, k$, $n = 1, 2, \ldots$, *and* z *of* I. CONCLUSION:

$$\sum_{n=1}^{\infty} \left(u_n^{(1)}(z) + u_n^{(2)}(z) + \ldots + u_n^{(k)}(z)\right)$$

converges non-uniformly over I.

Theorem 103. HYPOTHESES: (i) *for all values of* z *in the set* I, $|f(z)| < M$ *where* M *is constant*; (ii) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I. CONCLUSION: $\sum_{n=1}^{\infty} f(z)\, u_n(z)$ *converges uniformly over* I.

Theorem 104. HYPOTHESES: (i) $|f_i(z)| < M$, $i = 1, 2, \ldots$, z *of* I, *and* M *a constant*; (ii) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I; (iii) $u_n(z) \geqq 0$, $n = 1, 2, \ldots$, *for all values of* z *of* I. CONCLUSION: $\sum_{n=1}^{\infty} f_n(z)\, u_n(z)$ *converges uniformly over* I.

Theorem 105. HYPOTHESIS: $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I. CONCLUSION: *When any* $\epsilon > 0$ *is given it is possible to find an* M *such that* $|u_n(z)| < \epsilon$ *for all* z*'s of* I *simultaneously whenever* $n > M$.

Theorem 106. HYPOTHESIS: $\sum_{n=1}^{\infty} |u_n(z)|$ *converges uniformly over* I. CONCLUSION: $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I.

Theorem 107. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I; (ii) *there exists a second set* J *and a series of functions* $\sum_{n=1}^{\infty} U_n(z)$ *such that if* z' *is any point of* J, $U_n(z') = u_n(\bar{z})$, *where* z *is of* I. CONCLUSION: $\sum_{n=1}^{\infty} U_n(z')$ *converges uniformly over* J.

Theorem 108. HYPOTHESES: (i) $u_n(z) = v_n(z) + iw_n(z)$, *where* $v_n(z)$ *and* $w_n(z)$ *are real*; (ii) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I. CONCLUSION: $\sum_{n=1}^{\infty} v_n(z)$ *and* $\sum_{n=1}^{\infty} w_n(z)$ *both converge uniformly over* I.

Theorem 109. HYPOTHESIS: $s^{(i)}(z) \to f_i(z)$, $i = 1, 2, \ldots, k$, *uniformly over* I. CONCLUSION:

$$s_n^{(1)}(z)\, s_n^{(2)}(z) \ldots s_n^{(k)}(z) \to f_1(z) f_2(z) \ldots f_k(z)$$

uniformly over I.

Theorem 110. HYPOTHESIS: $s_n(z) \neq 0$ *approaches* $f(z) \neq 0$ *uniformly over* I. CONCLUSION: $\dfrac{1}{s_n(z)} \to \dfrac{1}{f(z)}$ *uniformly over* I.

§ 3. Continuity, &c.

Theorems 101 to 110 have been stated without proof. A good many other equally simple theorems relative to uniform convergence could be stated. Those given, however, are ample to illustrate the type. A slightly different kind of theorem, which requires more space and more extended thought, but which essentially depends upon hypotheses other than that of uniform convergence can be obtained by re-writing with proper modifications such theorems as 67 for uniformly convergent series. We proceed to less obvious and in this respect more important theorems.

Theorem 111. HYPOTHESES: (i) $u_1(z)$, $u_2(z)$, ..., *are continuous functions over a region of the complex plane which we denote by* I; (ii) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I *to* $f(z)$. CONCLUSION: $f(z)$ *is continuous at each point of* I.

PROOF: Let z_0 be any particular point of I. Let $\epsilon > 0$ be given and choose m so that

$$(1) \qquad |f(z) - s_m(z)| < \frac{\epsilon}{3}$$

simultaneously for every z of I. In particular,

$$(2) \qquad |s_m(z_0) - f(z_0)| < \frac{\epsilon}{3}.$$

Now $s_m(z)$ is continuous* since it is the sum of a fixed number of continuous functions. We choose δ, so that when $|z - z_0| < \delta$,

$$(3) \qquad |s_m(z) - s_m(z_0)| < \frac{\epsilon}{3}.$$

But

$$(4) \qquad |f(z) - f(z_0)| \leqq |f(z) - s_m(z)| + |s_m(z) - s_m(z_0)| + |s_m(z_0) - f(z_0)| < \epsilon,$$

thus establishing the continuity of $f(z)$.

It is to be noted that the region I may be one-dimensional, in particular, an interval on the axis of reals.

Uniform convergence is a sufficient condition for the continuity of the sum function but it is not a necessary condition. There follows an example of a series of continuous functions, which for any real value of the variable converges to zero and yet which does not converge uniformly over any interval whatever on the axis of reals.

As usual, to emphasize reality replace z by x, and let

$$\phi_n(x) = \sqrt{2e}\, e^{-n^2 \sin^4 \pi x} \sin^2 \pi x.$$

For any value of n, $\phi_n(x)$ has the period 1 and it is readily shown that within each period there are two maxima each of magnitude 1. Let

* The notion of continuity is assumed. See any book on the theory of functions.

$$s_n(x) = \phi_n(x) + \frac{1}{2!}\phi_n(2!\,x) + \ldots + \frac{1}{n!}\phi_n(n!\,x).$$

For every value of x this sequence approaches zero as n becomes infinite. We proceed to show this. Let an $\epsilon > 0$ be given. Take a particular x. Then, as we have just remarked, $\phi_n(x) \leqq 1$ for all values of n. Hence,

$$(5) \qquad \frac{1}{(k+1)!}\phi_n((k+1)!\,x) + \ldots + \frac{1}{n!}\phi_n(n!\,x) \leqq \frac{1}{(k+1)!} + \ldots + \frac{1}{n!} = \frac{1}{k!}\left[\frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \ldots + \frac{1}{(k+1)(k+2)\ldots n}\right]$$
$$< \frac{1}{k!}\left[\frac{1}{k+1} + \frac{1}{(k+1)^2} + \ldots + \frac{1}{(k+1)^{(n-k)}}\right] < \frac{1}{k!}\cdot\frac{1}{k} < \frac{\epsilon}{2}$$

if k is sufficiently great. Hold k fast. Now, from its definition for any value of x, $\phi_n(x) \to 0$. Consequently, $\phi_n(cx) \to 0$, where c is any constant. Hence we can choose an m such that when $n > m$,

$$(6) \qquad \left(\phi_n(x) + \frac{1}{2!}\phi_n(2!\,x) + \ldots + \frac{1}{k!}\phi_n(k!\,x)\right) < \frac{\epsilon}{2}.$$

From (5) and (6)

$$s_n(x) < \epsilon.$$

In other words, for any particular x, that is for every x,

$$s_n(x) \to 0.$$

However, $\frac{1}{m!}\phi_n(m!\,x) \to 0$ non-uniformly over any interval of length $\frac{1}{m!}$ since it always has within such an interval two maxima of magnitude $\frac{1}{m!}$. In order for $s_m(x)$, which is a sum of positive functions, to approach a limit uniformly, according to Theorem 101, each of the functions added to form it must approach its limit uniformly. Remembering this, select an interval of arbitrary length $\delta > 0$. When m is so great that $\frac{1}{m!} < \delta$, $\frac{1}{m!}\phi_n(m!\,x)$ does not converge uniformly in this interval. Consequently $s_n(x)$, which contains $\frac{1}{m!}\phi_n(m!\,x)$ when $n > m$, does not converge uni-

formly in the interval either, and this is what we wanted to show.

The transformation from the sequence to a series is immediate: $u_n(z) = s_n(z) - s_{n-1}(z)$.

Theorem 112. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(z) = f(z)$ *converges over a closed region* I; (ii) $u_n(z) \geqq 0$ *over* I; (iii) $u_n(z)$ *and* $f(z)$ *are continuous over* I. CONCLUSION: $\sum_{n=1} u_n(z)$ *converges uniformly over* I.

PROOF: Choose a particular point, z_0, of I and let any $\epsilon > 0$ be given. Give to $s_n(z)$ and $r_n(z)$ their usual meanings.

(7) $$r_n(z) = r_n(z_0) + s_n(z_0) - s_n(z) + f(z) - f(z_0).$$

Choose M so that $r_M(z_0) < \frac{1}{3}\epsilon$. Now choose a $\delta > 0$ so that when $|z - z_0| < \delta$,

$$|s_M(z) - s_M(z_0)| < \tfrac{1}{3}\epsilon \text{ and } |f(z) - f(z_0)| < \tfrac{1}{3}\epsilon.$$

This is possible because $s_M(z)$ and $f(z)$ are continuous. From (7), $r_M(z) < \epsilon$. But $r_n(z)$ does not increase as n increases; hence $r_n(z) < \epsilon$ when $n > M$ and $|z - z_0| < \delta$.

There then exists about z_0 which is any point of I a circle, c, such that $r_n(z) < \epsilon$ over that portion of I in c provided $n >$ a certain M. There is a lower limit * other than zero for the radii of these circles. Suppose this were not the case. Cover the region I by a square with sides parallel to the axes. Divide this into four equal squares by drawing lines bisecting its sides. Consider one of these smaller squares, including its boundary, which contains a portion of I where the radii have the lower limit zero. Divide it in like manner into four squares and proceed continually. We set up sections in both the real and pure imaginary numbers. We hence have a sequence of squares approaching a fixed point (number) as a limit. Take this point as the z_0 discussed above and we have a contradiction; because, about it there

* See chap. XII, § 2.

is a circle with radius greater than zero, over which $r_n(z) < \epsilon$ when $n > M$.

We complete the proof by covering I with squares, each of fixed side less than the lower limit for these radii. Choose M for each of these squares so that for all points of I in it $r_M(z) < \epsilon$. Call the largest of these, which are finite in number, M. When $n > M$, $r_n(z) < \epsilon$ for all points of I.

Theorem 113. HYPOTHESES: (i) $\sum_{n=1}^{\infty} v_n(z)$ *converges uniformly over* I; (ii) $v_n(z) \geqq 0$; (iii) $|u_n(z)| \leqq v_n(z)$, *whenever* z *belongs to* I. CONCLUSION: $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over* I.

PROOF: Let $\epsilon > 0$ be given and choose M so that whenever $n > M$ and $n' > M$ simultaneously, $\sum_{\nu=n'}^{n} v_\nu(z) < \epsilon$. Then

$$\left|\sum_{\nu=n'}^{n} u_\nu(z)\right| \leqq \sum_{\nu=n'}^{n} |u_\nu(z)| \leqq \sum_{\nu=n'}^{n} v_\nu(z) < \epsilon$$

for all z's of I simultaneously, which establishes the theorem.

Corollary 1. *The most important case under this theorem is where* $v_n(z) = M_n$, *a constant,* $n = 1, 2, \ldots$.

If $\sum_{n=1}^{\infty} |u_n(z)|$ converges uniformly over I, we say that $\sum_{n=1}^{\infty} u_n(z)$ converges absolutely uniformly over I. With this definition we have as a corollary to the theorem the following:

Corollary 2. *If a series converges absolutely uniformly over* I *it converges uniformly over* I.

Theorem 114. HYPOTHESES: (i)

$$\sum_{n=1}^{\infty} |u_n(z)|$$

converges uniformly over I; (ii) λ_n, $n = 1, 2, \ldots$, *are positive integers*; (iii) $\lambda_n \to \infty$. CONCLUSION:

$$\sum_{n=1}^{\infty} u_{\lambda_n}(z)$$

converges uniformly over I.

PROOF: Let $\bar{r}_m(z) = \sum_{\nu=m}^{\infty} |u_\nu(z)|$ and $\bar{\rho}_n(z) = \sum_{\nu=1}^{\infty} |u_{\lambda_\nu}(z)|$. Let n be given and choose m as small as possible, so that $\bar{r}_m$ will contain all terms in $\bar{\rho}_n$. With m so chosen, $\bar{\rho}_n(z) \leqq \bar{r}_m(z)$. When $n \to \infty$, $m \to \infty$, but $\bar{r}_m(z) \to 0$ uniformly over I, and hence $\bar{\rho}_n(z)$ does also; that is, $\sum_{n=1}^{\infty} |u_{\lambda_n}(z)|$ converges uniformly over I. Consequently $\sum_{n=1}^{\infty} u_{\lambda_n}(z)$ converges uniformly over I. See the previous theorem.

Theorem 115. HYPOTHESIS: *All the series*

$$\sum_{n=1}^{\infty} u_{\lambda_n}(z) \tag{8}$$

converge uniformly over I, *where* $\lambda_1, \lambda_2, \ldots$, *denote any arrangement whatever of the integers* $1, 2, \ldots$. CONCLUSION:

$$\sum_{n=1}^{\infty} |u_n(z)| \tag{9}$$

converges uniformly over I.

PROOF: (9) converges. Suppose that at z, some point of I, (9) diverged. Then, by a proper arrangement of the integers $1, 2, \ldots$, the series $\sum_{n=1}^{\infty} u_{\lambda_n}(\bar{z})$ could be made to diverge. (See Theorem 69, Cor. 2.) This is contrary to the hypothesis.

Now to show that (9) converges uniformly over I, again assume the contrary and also assume $u_n(z)$ real. Let, as usual, $s_n = \sum_{\nu=1}^{n} u_\nu(z)$ and $\sigma_n = \sum_{\nu=1}^{n} |u_\nu(z)|$. There exists

an $\epsilon > 0$ such that it is impossible to find an M such that for all values of $n, n' > M$, $|\sigma_n - \sigma_{n'}| < \epsilon$ simultaneously for every z in I. Let ϵ be such a number. There then are an infinite succession of number pairs, n_i and n_i', such that n_i and n_i' are both less than either n_{i+1} or n_{i+1}', and always $|\sigma_{n_i} - \sigma_{n_i'}| > \epsilon$ for some value of z of I. Denote a particular point where $|\sigma_{n_i} - \sigma_{n_i'}| > \epsilon$ by z_i, and when $u_n(z_i) \geqq 0$ denote it by $U_n(z_i)$ and when $u_n(z_i) < 0$ by $-V_n(z_i)$. Now, by a choice of the λ_n's rearrange the terms inside each of the parentheses $(s_n(z_i) - s_{n'}(z_i))$, so that all the $U_n(z_i)$'s come first and all the $V_n(z_i)$'s last. The $U_n(z_i)$'s plus the $V_n(z_i)$'s is greater than ϵ. Hence, at least the sum of the $U_n(z_i)$'s totals $\frac{\epsilon}{2}$ or the sum of the $V_n(z_i)$'s. Suppose, for example, that the sum of the $U(z_i)$'s does and denote it by

$$U_n^{(1)}(z_i) + U_n^{(2)}(z_i) + \ldots + U_n^{(k)}(z_i). \tag{10}$$

The series $\sum_{n=1}^{\infty} u_{\lambda_n}(z)$, where the arrangement is such as to contain each group of terms as (10), does not converge uniformly over I; for it is impossible to find an M but such that, with this arrangement, for some value of $\lambda_n > M$,

$$u_{\lambda_n} + \ldots + u_{\lambda_n + k} = U_n^{(1)} + \ldots + U_n^{(k)} > \frac{\epsilon}{2}$$

at some point.

Now suppose the terms of the series complex numbers

$$u_n(z) = w_n(z) + iv_n(z),$$

where $w_n(z)$ and $v_n(z)$ are real numbers. In order for a series $\sum_{n=1}^{\infty} u_{\lambda_n}(z)$ to converge uniformly, $\sum_{n=1}^{\infty} w_{\lambda_n}(z)$ and $\sum_{n=1}^{\infty} v_{\lambda_n}(z)$ must both do so. Consequently, under our theorem, $\sum_{n=1}^{\infty} |w_n(z)|$ and $\sum_{n=1}^{\infty} |v_n(z)|$, and hence

$$\sum_{n=1}^{\infty} (|w_n(z)| + |v_n(z)|)$$

converge uniformly. Hence, since

$$|u_n(z)| \leqq |w_n(z)| + |v_n(z)|,$$

$\sum_{n=1}^{\infty} |u_n(z)|$ converges uniformly over I by Theorem 113.

Theorem 116. HYPOTHESIS:

$$\sum_{n=1}^{\infty} u_n(z) \tag{11}$$

converges non-uniformly over a closed region, I, of the complex plane. CONCLUSION: *There exists at least one point, z_0, of I having the property that in every neighbourhood of z_0 portions of I exist over which* (11) *converges non-uniformly.*

PROOF: Consider a square, S_1, completely covering I with sides parallel to the axes. Divide S_1 into four equal squares, S_1', S_1'', S_1''', S_1'''', by lines through the centre parallel to the axes, each a closed region. Then (11) converges non-uniformly over that portion of I in at least one of these squares. Call this square S_2. Divide S_2, as previously, into four equal squares, S_2', S_2'', S_2''', S_2''''. Continue indefinitely. (11) converges non-uniformly over a portion of S_n. But S_n approaches a limiting point * which answers the requirements of z_0 of the theorem.

The language just used is pictorial. The facts, however, are, as usual, analytic. By the method of squares we set up a section in both the real and pure imaginary numbers, thus determining a complex number.

The theorem clearly might be stated and proved for one dimension where I is an interval on [redacted] curve, particularly the axis of reals.

Theorem 117. HYPOTHESES: (i) *$u_n(z)$ is real and defined at all points of I*; (ii) $M > u_n(z) \geqq u_{n+1}(z) > 0$; (iii) $\sum_{n=1}^{\infty} v_n(z)$ *converges uniformly over I.* CONCLUSION: $\sum_{n=1}^{\infty} u_n(z)\, v_n(z)$ *converges uniformly over I.*

* See chap. XII, § 2; also [redacted] proof of Theorem 135.

PROOF: Apply Theorem 28,

$$\left|\sum_{\nu=n}^{n+p} u_\nu(z) v_\nu(z)\right| \leqq u_n(z)\, G_n(z) < M G_n(z),$$

where $G_n(z)$ is the greatest of the sums

$$\left|\sum_{\nu=n}^{m} v_\nu(z\right|, \quad 0 \leqq m \leqq n+p,$$

at the point z. But $G_n(z) \to 0$ uniformly. The theorem follows.

Theorem 118. HYPOTHESES: (i) *$u_n(z)$ and $v_n(z)$ are defined at all points of I*; (ii) *$u_n(z) < M$, independent of z and n*; (iii) $u_n(z) \geqq u_{n+1}(z)$; (iv) *$u_n(z) \to$ [redacted] uniformly over I*;

$$\text{(v)} \quad \left|\sum_{\nu=n}^{n+p} v_\nu(z)\right| < G,$$

independent of z and n and $p > 0$. CONCLUSION:

$$\sum_{n=1}^{\infty} u_n(z)\, v_n(z)$$

converges uniformly over I.

PROOF: Proceed as in the previous theorem. Details are omitted.

Theorem 119. HYPOTHESES: (i) *$a_n(z)$ and $b_n(z)$, $n = 1, 2, \ldots$, are defined over a set, I*;

$$\text{(ii)} \quad \sum_{n=1}^{\infty} a_n(z) \ \textit{and} \ \sum_{n=1}^{\infty} |b_n(z) - b_{n+1}(z)|$$

converge uniformly over I; (iii) *$|b_n(z)| \leqq c_n$ independent of z.*

CONCLUSION: $\sum_{n=1}^{\infty} a_n(z)\, b_n(z)$ *converges uniformly over I.*

PROOF: As a result of hypothesis (ii), $\sum_{n=1}^{\infty} (b_n(z) - b_{n+1}(z))$ converges uniformly over I. Hence there exists an N, such that when $n > N$,

$$\left|\sum_{n=N}^{n-1} (b_n - b_{n+1})\right| < 1;$$

that is $|b_N - b_n| < 1$; and consequently,

$$|b_n| < 1 + |b_N| \leqq 1 + c_N.$$

Let c equal the maximum of $c_1, \ldots, c_N$. Then, for any value of n, $|b_n| \leqq 1 + c = C$.

Choose $a_1(z)$, so that $\sum_{n=1}^{\infty} a_n(z) \equiv 0$. In this there is no loss of generality. Simply replace $a_1(z)$ by

$$\bar{a}_1(z) \equiv a_1(z) - \sum_{n=1}^{\infty} a_n(z),$$

and let $s_n = \sum_{\nu=1}^{n} a_\nu$.

$$\sum_{\nu=n+1}^{n+p} a_\nu b_\nu = \sum_{\nu=n+1}^{n+p} (s_\nu - s_{\nu-1}) b_\nu$$

$$= \sum_{\nu=n+1}^{n+p-1} s_\nu (b_\nu - b_{\nu+1}) - s_n b_{n+1} + s_{n+p} b_{n+p}.$$

Now, let $\delta > 0$ be given and choose $M > N$, so that, when $n > M$, $|s_n(z)| < \delta$ uniformly. Moreover, by Theorem 96,

$$\sum_{\nu=n+1}^{n+p-1} |b_\nu(z) - b_{\nu+1}(z)| < 1$$

if $p > 1$ and n is greater than sufficiently great M. We suppose M chosen great enough. Then

$$\left| \sum_{\nu=n+1}^{n+p} a_\nu(z) b_\nu(z) \right| < \delta + \delta C + \delta C < \epsilon$$

if $\delta < \dfrac{\epsilon}{3C}$.

This completes the proof.

Theorem 120. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n(z)$ *converges uniformly over* I; (ii) $\sum_{n=1}^{\infty} |b_n(z) - b_{n+1}(z)| \leqq G$ *for all values of* z *of* I; (iii) $|b_1(z)| \leqq b$ *when* z *is of* I. CONCLUSION:

$$\sum_{n=1}^{\infty} a_n(z) b_n(z)$$

converges uniformly over I.

PROOF: Let $g = b + G$,

$$|b_1 - b_n| = \left| \sum_{\nu=1}^{n-1} (b_\nu - b_{\nu+1}) \right| \leqq \sum_{\nu=1}^{n-1} |b_\nu - b_{\nu+1}| \leqq G.$$

Hence $|b_n| \leqq g$. Next assume $\sum_{n=1}^{\infty} a_n(z) \equiv 0$ in which again there is no loss of generality. Then

$$\sum_{\nu=u}^{v} a_\nu b_\nu = \sum_{\nu=u}^{v} (s_\nu - s_{\nu-1}) b_\nu$$

$$= \sum_{\nu=u}^{v} s_\nu (b_\nu - b_{\nu+1}) - s_{u-1} b_u + s_v b_{v+1}.$$

Choose M so that when $n > M$, $|s_n(z)| < \delta$ uniformly over I; and let $n > M$. Then

$$\left| \sum_{\nu=u}^{v} a_\nu b_\nu \right| < \sum_{\nu=u}^{\infty} |s_\nu| \, |b_\nu - b_{\nu+1}| + \delta g + \delta g$$

$$< \delta \sum_{\nu=1}^{\infty} |b_\nu - b_{\nu+1}| + 2\delta g \leqq \delta G + 2\delta g.$$

Now if any $\epsilon > 0$ is given, we choose $\delta < \dfrac{\epsilon}{3g}$ and also $< \dfrac{\epsilon}{3G}$.

Then
$$\left| \sum_{\nu=u}^{v} a_\nu(z) b_\nu(z) \right| < \epsilon;$$

establishing the theorem.

Theorem 121. HYPOTHESES: (i) $\sum_{n=1}^{\infty} \phi_n(z)$ *converges uniformly over a set of points,* I, *to* $\phi(z)$; (ii) $\phi_n(z) \to \Phi_n$ *when* z *behaves in a prescribed manner on* I; *for convenience of notation we say when* $z \to z_0$ *a limiting point* * *of* I. CONCLUSIONS: (i) $\sum_{n=1}^{\infty} \Phi_n = \Phi$ *converges*; (ii) $\sum_{n=1}^{\infty} \phi_n(z) \to \Phi$ *when* $z \to z_0$.

PROOF: Let $\sum_{1}^{n} \phi_n(z) = s_n(z)$ and $\sum_{1}^{n} \Phi_n = S_n$, also let

* See chap. XII, § 2.

an $\epsilon > 0$ be given. Choose M so large that, when $n \geqq M$ and $n' \geqq M$,

(12) $$|s_n(z) - s_{n'}(z)| < \tfrac{1}{3}\epsilon$$

for all z's of I simultaneously. Then, since for any value of n when $z \to z_0$, $s_n(z) \to S_n$,

(13) $$|S_n - S_{n'}| \leqq \tfrac{1}{3}\epsilon.$$

We immediately draw conclusion (i). (See Theorem 16.) The proof of (ii) is similar to the continuity proof in Theorem 111. Choose δ so that $|s_M(z) - S_M| < \frac{\epsilon}{3}$ when $|z - z_0| < \delta$. From (12) and (13), $|\phi(z) - s_M(z)| \leqq \frac{\epsilon}{3}$ and $|\Phi - S_M| \leqq \frac{\epsilon}{3}$. Combining these three inequalities, $|\phi(z) - \Phi| < \epsilon$, establishing conclusion (ii).

In case z does not approach z_0, but becomes infinite on I or behaves in other prescribed manner, a slight modification of the wording of the above proof is necessary. In case z does approach z_0, a limiting point of I but not belonging to I, the series converges uniformly over the set of points composed of the points of I and z_0. This follows from the inequalities above written,

$$|s_n(z) - s_{n'}(z)| < \tfrac{1}{3}\epsilon \text{ and } |S_n - S_{n'}| < \tfrac{1}{3}\epsilon.$$

Conclusion (ii) could then be made to follow from Theorem 111.

Theorem 122. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to* $f(x)$ *over an interval of the x-axis defined by the inequalities* $a \leqq x \leqq b$; (ii) $\int_a^b u_n(x)\,dx$* *exists.* CONCLUSIONS:

(i) $\int_a^b f(x)\,dx$ *exists and* (ii) $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$

PROOF: (1) Let an $\epsilon > 0$ be given. Divide the interval (a, b) into n portions $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$. Let O_k be the

* Riemann integrals.

oscillation * of $f(x)$ in Δx_k. Let $f(x) = s_n(x) + r_n(x)$, where

$$s_n(x) = \sum_{\nu=1}^{n} u_\nu(x),$$

and let ${}_nO_k$ and ${}_n\omega_k$ be the oscillations of $s_n(x)$ and $r_n(x)$, respectively, in Δx_k.

Then $O_k \leqq {}_nO_k + {}_n\omega_k$ and consequently

$$\Sigma O_k \Delta x_k \leqq \Sigma\, {}_nO_k \Delta x_k + \Sigma\, {}_n\omega_k \Delta x_k,$$

where the summation is extended to all the subdivisions of (a, b).

Now choose an $\eta > 0$ and take n so large that $|r_n(x)| < \eta$ uniformly over (a, b). Then ${}_n\omega_k < 2\eta$. Then

$$\Sigma\, {}_n\omega_k \Delta x_k < 2\eta \Sigma \Delta x_k = 2\eta(b - a).$$

s_n is integrable, being the sum of a finite number of integrable functions. Consequently, if each Δx_k is sufficiently small,

$$\Sigma\, {}_nO_k \Delta x_k < \eta.$$

Then $$\Sigma O_k \Delta x_k < \eta(2(b - a) + 1) < \epsilon,$$

when $$\eta < \frac{\epsilon}{2(b - a) + 1}.$$

$f(x)$ is consequently integrable, and hence since $s_n(x)$ is integrable $r_n(x)$ is integrable also.

(2) $$\int_a^b f(x)\,dx = \int_a^b s_n(x)\,dx + \int_a^b r_n(x)\,dx$$
$$= \sum_{n=1}^{n} \int_a^b u_n(x)\,dx + \int_a^b r_n(x)\,dx.$$

Choose $\eta > 0$ and M so that, when $n > M$, $|r_n(x)| < \eta$ simultaneously for all x's of (a, b). Then

$$\left| \int_a^b f(x)\,dx - \sum_{n=1}^{n} \int_a^b u_n(x)\,dx \right| < \eta(b - a),$$

which is arbitrarily small. This completes the proof by the definition of convergence.

* See, for example, Pierpont, *The Theory of Functions of Real Variables*, vol. i, p. 341.

Theorem 123. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly to* $f(z)$ *over a curve,* c, *of the complex plane, of length less than* A; (ii) $u_n(z)$ *is integrable over* c. CONCLUSION: $f(z)$ *is integrable over* c *and*

$$\int_c f(z)\,dz = \sum_{n=1}^{\infty} \int_c u_n(z)\,dz.$$

PROOF: Let $z = x + yi$, where x and y are real, and let

$$u_n(z) = w_n(x, y) + i v_n(x, y),$$

where w_n and v_n are real. Similarly let

$$f(z) = w(x, y) + i v(x, y).$$

Then $$\int_c u_n(z)\,dz = \int_c w_n\,dx - v_n\,dy + i \int_c v_n\,dx + w_n\,dy$$

and, if all functions are integrable,

$$\int_c f(z)\,dz = \int_c w\,dx - v\,dy + i \int_c v\,dx + w\,dy.$$

The establishment of this theorem from now on involves much repetition from the proof of the previous theorem. It is omitted.

A well-known example to illustrate the value of the last two theorems is the following:

$$u_n(x) = s_n(x) - s_{n-1}(x), \quad n > 1,$$
$$u_1(x) = s_1(x),$$

where $$s_n(x) = nxe^{-nx^2}.$$

It is readily shown that $s_n(x) \to 0$ for every real value of x; that is for real values of x, $f(x) \equiv 0$. Hence

$$\int_0^1 f(x)\,dx = 0.$$

But $$\int_0^1 s_n(x)\,dx = \tfrac{1}{2}[1 - e^{-n}] \to \tfrac{1}{2}.$$

The fact that these two results are different is interesting. Clearly $s_n(x)$ cannot approach $f(x)$ uniformly over the interval $0 \leqq x \leqq 1$, else they would be the same.

Theorem 124. HYPOTHESES: (i) $u_n(z)$, $n = 1, 2, \ldots$, *are single-valued continuous functions of* z *at all points of a region* S, *any two points of which can be connected by a continuous curve lying wholly within* S *and of length less than* A; (ii) *there exists a point,* z_0, *of* S *such that*

$$\sum_{n=1}^{\infty} u_n(z_0)$$

converges; (iii) $u_n'(z)$, $n = 1, 2, \ldots$, *where the accent denotes the derivative, exists at every point of* S; (iv) $\sum_{n=1}^{\infty} u_n'(z)$ *converges uniformly over* S. CONCLUSIONS: (i) $\sum_{n=1}^{\infty} u_n(z)$ *converges over* S *to a function* $f(z)$, *and* (ii) $f'(z) = \sum_{n=1}^{\infty} u_n'(z)$.

PROOF: Let $\phi(z) = \sum_{n=1}^{\infty} u_n'(z)$. Let c be a curve from z_0 to z of length less than A and integrate along c. By Theorem 123,

$$\int_{z_0}^{z} \phi(z)\,dz = \sum_{n=1}^{\infty} \int_{z_0}^{z} u_n'(z)\,dz = \sum_{n=1}^{\infty} [u_n(z) - u_n(z_0)].$$

Denoting $\sum_{n=1}^{\infty} u_n(z_0)$ by $f(z_0)$, we have

$$\int_{z_0}^{z} \phi(z)\,dz + f(z_0) = \sum_{n=1}^{\infty} u_n(z), \tag{14}$$

that is $\sum_{n=1}^{\infty} u_n(z)$ converges. Denote its sum by $f(z)$. Then differentiating (14) and replacing $\sum_{n=1}^{\infty} u_n(z)$ by $f(z)$ we have

$$f'(z) = \phi(z) = \sum_{n=1}^{\infty} u_n'(z);$$

completing the proof.

§ 4. Uniform convergence of series in several variables.

Definition 22. *Let* $\sum_{n=1}^{\infty} u_n(z_1, \ldots, z_k)$ *converge to* $f(z_1, \ldots, z_k)$ *over a set of points,* $(z_1, \ldots, z_k)$, *which we denote by* I. *Let*

$$r_n = \sum_{\nu=n+1}^{\infty} u_\nu(z_1, \ldots, z_k).$$

If, given any $\epsilon > 0$, *it is possible to choose an* M, *so that* $|r_n| < \epsilon$ *when* $n > M$, *for all points of* I *simultaneously, then*

$$\sum_{n=1}^{\infty} u_n(z_1, \ldots, z_k)$$

is said to converge uniformly over I.

The theorems of this chapter usually can be generalized to k variables.

§ 5. Relative uniform convergence.

Definition 23. *Suppose that* $u_n(z)$, $n = 1, 2, \ldots$, *is defined at each point of a set* I, *and that a function* $\sigma(z)$ *is also defined at each point of* I. $\sum_{n=1}^{\infty} u_n(z)$ *is said to converge over* I *uniformly relative to* $\sigma(z)$, *if, given any* $\epsilon > 0$, *there exists an* M *such that when* $n > M$,

$$\left| \sum_{\nu=n}^{\infty} u_\nu(z) \right| \leqq \epsilon |\sigma(z)|$$

at each point of I *simultaneously.*

Clearly uniform convergence is a special case of relative uniform convergence, $\sigma(z) \equiv 1$.

Generalization of Theorems 96 and 97 is immediate.

The following statements are readily proved. Although distinct theorems, they are put under one theorem number and are written in abbreviated form.

Theorem 125. *If*

$$(1) \qquad \sum_{n=1}^{\infty} u_n(z)$$

converges uniformly over I *relative to* $\sigma(z)$, *it converges uniformly over* I *relative to* $\tau(z)$ *if* $|\tau(z)| \geqq |\sigma(z)|$.

Uniform convergence relative to $\sigma(z)$ *and uniform convergence relative to* $a\sigma(z)$, $a \neq 0$ *a constant, are equivalent.*

If (1) *converges uniformly over* I *relative to* 0, *it converges uniformly over* I *relative to every function.*

If $|\sigma(z)| < M$, *independent of* z, *and if* (1) *converges uniformly over* I *relative to* $\sigma(z)$, *then it converges uniformly over* I.

Uniform convergence of $\sum_{n=1}^{\infty} \mu u_n(z)$ *relative to* $\mu\sigma$ *implies uniform convergence of* $\sum_{n=1}^{\infty} u_n(z)$ *relative to* σ.

Theorem 126. HYPOTHESIS: *There exists no number* M *such that* $|\sigma(z)| < M$ *for all* z*'s of* I. CONCLUSION: *There exist series converging over* I *uniformly relative to* $\sigma(z)$ *but not uniformly.*

PROOF: We cite the example

$$\sum_{n=1}^{n} u_n(z) = s_n(z) = a_n \sigma(z),$$

where $a_n \leqq a_{n+1}$ and $a_n \to 1$.

Many of the simpler theorems on uniform convergence can be generalized to relative uniform convergence without difficulty. For example, we cite numbers 100, 101, and 102. The following theorems are given, in part at least, independent proof.

Theorem 127. HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(x) = \phi(x)$ *converges over the interval* $a \leqq x \leqq b$, *uniformly relative to* $\sigma(x)$; (ii) $\int_a^b u_n(x)\,dx$ *exists for all values of* n; (iii) $\int_a^b |\sigma(x)|\,dx$ *exists.* CONCLUSION: $\phi(x)$ *is integrable from* a *to* b *and*

$$\int_a^b \phi(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

PROOF: The existence proof follows so closely that under Theorem 122 that it is omitted. The last conclusion is also readily proved thus:

$$\int_a^b \phi(x)\,dx = \sum_{n=1}^{n-1}\int_a^b u_n(x)\,dx + \int_a^b \sum_{n=n}^{\infty} u_n(x)\,dx.$$

But $$\left|\int_a^b \sum_{n=n}^{\infty} u_n(x)\,dx\right| \leqq \epsilon \int_a^b |\sigma(x)|\,dx,$$

which last can be made as small as desired by taking ϵ small enough. It in turn can be made as small as desired by taking n greater than the corresponding M.

Theorem 128. HYPOTHESIS: $\lim\limits_{n\to\infty} c_n a^n = 0$, $a \neq 0$. CONCLUSION: $\sum\limits_{n=0}^{\infty} c_n x^n$ *converges uniformly relative to* $\dfrac{1}{1-\dfrac{x}{a}}$ *over the interval* $0 \leqq x < a$.

PROOF: Suppose $c_n a^n < \epsilon$ when $n > M$, and let n be greater than M, then

$$|c_n x^n + c_{n+1}x^{n+1} + \dots| = \left|c_n a^n\left(\frac{x}{a}\right)^n + c_{n+1}a^{n+1}\left(\frac{x}{a}\right)^{n+1} + \dots\right|$$

$$\leqq \epsilon\left|\left(\frac{x}{a}\right)^n + \left(\frac{x}{a}\right)^{n+1} + \dots\right| = \epsilon\left(\frac{x}{a}\right)^n \cdot \frac{1}{1-\dfrac{x}{a}} \leqq \epsilon\,\frac{1}{1-\dfrac{x}{a}}.$$

EXERCISES

The first corollary to Theorem 113 is frequently referred to as the M-test.

194–199. Examine the following series for uniform convergence. Consider all intervals on the axis of reals.

$$1 + xa + \frac{x(x-1)}{2!}a^2 + \dots;$$

$$e^x \sin x + e^{2x}\sin 2x + e^{3x}\sin 3x + \dots;$$

$$1 + x + x^2 + \dots;$$

$$x + \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^4}{4} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^6}{6} + \dots;$$

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \dots;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2}e^{-\frac{x^2}{(n+1)^2}}\right).$$

200. Prove the following theorem: HYPOTHESES: (i) $u_k(x)$ remains finite for all x of I; (ii) $\left|\dfrac{u_{n+1}(x)}{u_n(x)}\right| \leqq r < 1$ for all $n \geqq K$ and for all x of I. CONCLUSION: $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly over I.

201. Generalize other of the tests for convergence of series with positive terms to uniform convergence.

202. Generalize Theorem 66 to uniform convergence.

203. Discuss uniform convergence of

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \dots$$

for real values of x. Denoting the function represented by $\zeta(x)$, draw graph of $y = \zeta(x)$.

204. Prove the following theorem. HYPOTHESES: (i) n and m are positive integers; (ii) $s_n(m) \to f(m)$ uniformly for all m when $n \to \infty$; (iii) $s_n(m) \to S_n$ when $m \to \infty$. CONCLUSIONS: (i) $\lim\limits_{n\to\infty} f(m)$ exists. Call it A; (ii) $\lim\limits_{m\to\infty} S_n$ exists. Call it B; (iii) $A = B$.

205. Prove: $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = 1 + x + \dfrac{x^2}{2!} + \dots.$

206. Given

$$\cos mx = \cos^m x - \frac{m(m-1)}{1.2}\cos^{m-2} x \sin^2 x + \dots + (-1)^{m-1}\sin^m x,$$

prove: $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

207. Prove the following theorem: HYPOTHESES: (i) $s_n(x)$ and $f(x)$ are both continuous over $a \leqq x \leqq b$; (ii) $s_n(x) \to f(x)$ for all rational values of x on $a \leqq x \leqq b$;

$$\text{(iii)}\quad |s_n(x') - s_n(x)| < M,$$

where M is fixed. CONCLUSION: $s_n(x) \to f(x)$ uniformly over $a \leqq x \leqq b$.

208. Prove the following theorem: HYPOTHESES: (i) $u_n(x)$ is uniformly continuous* when $a < x < b$; (ii) $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly when $a < x < b$. CONCLUSION: The function $f(x)$ represented by the series approaches a limit as x approaches a or b.

209. Is the following theorem true or false? HYPOTHESES: (i) $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly, $a < x \leqq b$; (ii) $\sum_{n=1}^{\infty} u_n(a)$ converges. CONCLUSION: $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly, $a \leqq x \leqq b$.

210. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly over any closed sub-interval whatsoever, $\alpha \leqq x \leqq \beta$, of $a < x < b$, and if $u_n(x)$ is continuous over $a \leqq x \leqq b$, does $\sum_{n=1}^{\infty} u_n(x)$ necessarily converge when $x = a$? If it does, is the function represented by the series necessarily continuous at a?

HINT: See chapters on power series and Fourier series.

211. Let $\sum_{n=1}^{n} u_n(z) = s_n(z)$ and let the zeros of $s_n(z)$ for all values of n be marked in the complex plane. If these zeros have $z = \alpha$ as a limiting point,† prove that the series $\sum_{n=1}^{\infty} u_n(z)$ has α as a zero, provided that it converges uniformly over a region having α as an interior point.

* See, for example, Pierpont, *The Theory of Functions of Real Variables*, vol. i, p. 215.

† See chap. XII, § 2.

212. The series, where $s_n = nxe^{-nx^2}$, converges non-uniformly over any interval on the axis of reals including the origin. Draw graphs of $y = nxe^{-nx^2}$ for $n = 1, 4, 9, 16, 25$. Discuss the behaviour of the graph when $n \to \infty$. Integrate $s_n(x)$ from 0 to 1 and take the limit of this result when $n \to \infty$. Discuss the continuity of $\lim_{n \to \infty} s_n(x)$. Integrate this limit function from 0 to 1.

213–219. Treat as in the above exercise the series, where

$$s_n = \frac{nx}{1 + n^2x^2}, \quad s_n = x^n, \quad s_n = \frac{x^n}{1 + x^{2n}},$$

$$s_n = \frac{n\sqrt{x}}{1 + n^2x^3}, \quad s_n = \frac{n^2x}{1 + n^3x^3}, \quad s_n = \frac{nx}{1 + n^3x^6}, \quad s_n = nx^2e^{-nx^3}.$$

220. Discuss uniform convergence of

$$\sum_{n=1}^{\infty} a_n x \left(\frac{\sin nx}{nx}\right)^2:$$

(i) when $a_n \to 0$; (ii) when $a_n = 1$.

221. Discuss uniform convergence of a power series,

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots .$$

222. Prove the following theorem: HYPOTHESES:

$$\text{(i)}\quad \sum_{n=1}^{\infty} u_n(x)$$

converges at all points of the interval $a \leqq x \leqq b$;

$$\text{(ii)}\quad \left|\sum_{n=1}^{\infty} u_n'(x)\right| < M$$

at all points of $a \leqq x \leqq b$. CONCLUSION:

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly over $a \leqq x \leqq b$.

223–227. Differentiate the following series term by term and discuss:

$$\sum_{n=0}^{\infty}\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right],$$

$$\sum_{n=1}^{\infty}\left[\frac{\arctan x\sqrt{n}}{\sqrt{n}}-\frac{\arctan x\sqrt{n+1}}{\sqrt{n+1}}\right],$$

$$\sum_{n=1}^{\infty}\left[\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}}\right],$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{2n}\log(1+n^2x^2)-\frac{1}{2(n+1)}\log\left(1+(n+1)^2x^2\right)\right],$$

$$\sum_{n=1}^{\infty}\left[\frac{2nx}{e^{nx^2}}-\frac{2n^2x^3}{e^{nx^3}}-\frac{2(n+1)x}{e^{(n+1)x^2}}+\frac{2(n+1)^2x^3}{e^{(n+1)x^3}}\right].$$

CHAPTER X

CONTINUITY AND INTEGRABILITY. QUASI-, SUB-, AND INFRA-UNIFORM CONVERGENCE

§ 1. Quasi-uniform convergence.

One of the most important properties of uniformly convergent series is that the sum of a uniformly convergent series of continuous functions is itself a continuous function (Theorem 111). A type of convergence somewhat more general than uniform convergence but which has this important property is the following.

Definition 24. *The series* $\sum_{n=1}^{\infty} u_n(x)$, *which converges at every point of a given set, I, to* $f(z)$, *is said to converge quasi-uniformly over I, if corresponding to any* $\epsilon > 0$, *there exist an infinite number of values of n such that*

$$|r_n(z)| = \left|\sum_{\nu=n+1}^{\infty} u_\nu(z)\right| < \epsilon$$

simultaneously for all points of I.

Theorem 129. HYPOTHESIS:

(1) $$\sum_{n=1}^{\infty} u_n(z)$$

converges quasi-uniformly over I. CONCLUSION: *there exists at least one set of positive integers,* $n_1 < n_2 < n_3 < \dots$, *such that the series*

(2) $$\sum_{j=1}^{\infty}(u_{n_j+1}+u_{n_j+2}+\dots+u_{n_{j+1}})$$

converges uniformly over I.

PROOF: Let $\epsilon_1, \epsilon_2, \dots,$ be a sequence of continually diminishing positive numbers which converges to zero. An integer

n_1 can be found such that $|r_{n_1}(z)| < \epsilon_1$ for all z's of I simultaneously. Choose n_1 and hold it fixed. Now choose $n_2 > n_1$ and such that $|r_{n_2}(z)| < \epsilon_2$ simultaneously for all z's of I. Hold n_2 fixed and choose $n_3 > n_2$ so that $|r_{n_3}(z)| < \epsilon_3$, &c. This choice of $n_1, n_2, n_3, \ldots$ causes (2) to be uniformly convergent by the definition of uniform convergence.

The converse of this theorem is immediate; namely, that a uniformly convergent series can be replaced by one that is only quasi-uniformly convergent. Replace, for example, $u_n(z)$ by $U_n^{(1)}(z) + U_n^{(2)}(z) + \ldots + U_n^{(k)}(z)$, $k > 1$, and where $U_n^{(k)}(z) \to 0$ but not uniformly over I.

Theorem 130. HYPOTHESIS:

(1) $$u_1(z),\ u_2(z)\ \ldots,$$

are continuous functions over a region, I, of the complex plane.

(2) $$\sum_{n=1}^{\infty} u_n(z)$$

converges quasi-uniformly over I to $f(z)$. CONCLUSION: *$f(z)$ is continuous over I.*

PROOF: Group the terms of the series in parenthesis as suggested in the previous theorem. The resulting series converges uniformly to $f(z)$ which consequently is continuous.

§ 2. Necessary and sufficient condition for continuity.

Theorem 131. *Let*

$$f(z) = \sum_{n=1}^{\infty} u_n(z) \quad \textit{and} \quad r_n(z) = \sum_{\nu=n+1}^{\infty} u_\nu(z),$$

where every $u_n(z)$ is defined in a neighbourhood, real or complex, of z_0 and is continuous at z_0. A necessary and sufficient condition that $f(z)$ be continuous at z_0 is, that corresponding to each $\epsilon > 0$ a number n_1 can be found, such that for each value of $n > n_1$ a neighbourhood of z_0, defined by $|z - z_0| \leqq \delta$, $\delta > 0$, can be found at every point of which $|r_n(z)| < \epsilon$. The number δ is in general dependent on n.

(i) To prove the condition necessary:

By hypothesis $f(z)$ is continuous at z_0; and hence,

(1) $$|f(z) - f(z_0)| = |s_n(z) + r_n(z) - s_n(z_0) - r_n(z_0)| < \frac{\epsilon}{3}$$

when $|z - z_0| \leqq \delta_1$. Now choose n so that

(2) $$r_n(z_0)| < \frac{\epsilon}{3}$$

for it and all larger values of n. Due to the continuity of $u_n(z)$,

(3) $$|s_n(z_0) - s_n(z)| < \frac{\epsilon}{3}$$

when $|z - z_0| \leqq \delta_2$. Let δ be the smaller of δ_1 and δ_2 and combine (1), (2), and (3). We get $|r_n(z)| < \epsilon$.

(ii) To prove the condition sufficient:

For a properly chosen n and $\delta > 0$, when $|z - z_0| \leqq \delta$,

$$|r_n(z)| = |s_n(z) - f(z)| < \frac{\epsilon}{3},$$

$$|r_n(z_0)| = |s_n(z_0) - f(z_0)| < \frac{\epsilon}{3},$$

$$|s_n(z_0) - s_n(z)| < \frac{\epsilon}{3}.$$

Combining, $|f(z) - f(z_0)| < \epsilon$, completing the proof.

In case the δ of the theorem is independent of n, the point z_0 is called a point of uniform convergence. In the case of reals we may have uniform convergence on the left and not on the right and vice versa. In general the uniform convergence may be over any region having z_0 as a boundary point.

Statements of this theorem similar to Theorems **96** and **97** for uniform convergence are possible.

§ 3. Sub-uniform convergence.

Definition 25. *Given that*

(1) $$\sum_{n=1}^{\infty} u_n(x)$$

converges at all points of

(2) $$a \leqq x \leqq b$$

to $f(x)$, and that, as usual,

$$r_n(x) = f(x) - \sum_{\nu=1}^{n} u_\nu(x) = f(x) - s_n(x);$$

when to any $\epsilon > 0$ and any $M > 0$ there corresponds a method of division of (2) into a finite number of closed sub-intervals, $p_1, \dots, p_{m(\epsilon)}$, so that there is a value of $n, n(p_j) > M$, such that $|r_n(x)| < \epsilon$ so long as x is in p_j and $n = n(p_j)$, (1) is said to converge sub-uniformly over (2).

Theorem 132. *If $u_1(x), u_2(x), \dots, u_n(x) \dots$, are continuous over (2), a necessary and sufficient condition that $f(x)$ be continuous over (2) is that (1) converge sub-uniformly over (2).*

PROOF: I. Sufficient. Each point is at all times a point of one of the intervals, $p_1, \dots, p_{m(\epsilon)}$, of Definition 25. Each point is then surrounded by an interval over which $|r_n(x)| < \epsilon$ with the proper choice of $n > M$ or is the end point of such an interval. Call a particular point x_0. Group the terms in parentheses as in Theorem 129. In case x_0 is always an interior point of one of the intervals, continuity follows by Theorem 131. In case x_0 is a or b, we are interested in continuity to the right or left only, which also follows.

With a proper method of division, the case of x_0 being an end point of an interval other than a or b need never occur. If by a particular method of subdivision x_0 is an end point of two intervals; suppose that $|r_{\overline{M}}(x)| < \epsilon$ for the interval to the left, p_k, and that $|r_{\overline{\overline{M}}}(x)| < \epsilon$ for the interval to the right, p_{k+1}. For definiteness, suppose $\overline{\overline{M}} > \overline{M}$. $\overline{\overline{M}}$ and $\overline{M}$ are particular numbers. As $u_{\overline{M}}(x) + u_{\overline{M}+1}(x) + \dots + u_{\overline{\overline{M}}-1}(x)$ is continuous, by taking an interval to the left of x_0 small enough we can make $|r_{M}(x)|$ within it as close to $|r_{\overline{\overline{M}}}(x_0)|$, a constant, as we like; in particular less than ϵ since $|r_{\overline{\overline{M}}}(x_0)| < \epsilon$. We include such an interval to the left with p_{k+1}, and x_0 is not an end point of any interval.

Continuity at x_0, which is any point, implies continuity throughout the interval.

II. We shall now prove the condition necessary.

By Theorem 131, given any $\epsilon > 0$ there is an M such that when $n > M$ an interval, of length $2\delta_n(x)$, has x as mid-point; such that $|r_n(\bar{x})| < \epsilon$ so long as $\bar{x}$ is of the interval. In case x is a (or b) a similar interval of length δ_n extends from a (or b) toward the interior of (2). For all values of $n > M$, x held fast, $\delta_n(x)$ has a superior limit* (possibly infinite). Denote this by $S_M(x)$. Now, as a function of x, $S_M(x)$ has an inferior limit η. We shall prove $\eta > 0$. Suppose the contrary. Divide (2) into two closed intervals of equal length. $S_M(x)$ has an inferior limit of 0 in one of these intervals. Continue this process. By repeated divisions into halves we arrive at a point, ξ, of (2) such that there do not exist δ_n's greater than 0, $n > M$, such that $|r_n(x)| < \epsilon$ when $|x - \xi| < \delta_n$. By Theorem 131 this means that $f(x)$ is discontinuous at ξ; a contradiction.

Now, knowing $\eta > 0$, choose a fixed ζ, such that $0 < \zeta < \eta$. Let $n_0 > M$ be such that $\delta_{n_0}(a) < \zeta$. Suppose that the right-hand extremity of the interval of length $\delta_{n_0}(a)$ measured from a is x_1; that is, $|r_{n_0}(x)| < \epsilon$ when $a \leqq x < x_1$, but $|r_{n_0}(x_1)| = \epsilon$. Choose $n_1 > M$, such that $\delta_{n_1}(x_1) > \zeta$. Suppose the right-hand extremity of the interval of length $2\delta_{n_1}(x_1)$, with x_1 as mid-point, to be x_2, &c. With a finite number of steps we reach b. Where two or more intervals overlap, assign the overlapping part to one of the intervals only. For example, choose a division-point X_1 between a and x_1 such that $|r_{n_0}(x)| < \epsilon$ when $a \leqq x \leqq X_1$ and $|r_{n_1}(x)| < \epsilon$ when $X_1 \leqq x \leqq x_1$, &c. The subdivision of (2) made by $a, X_1, X_2, \dots, b$, fulfils the requirements of the theorem; and proof is complete.

It is to be noticed that the discussion of sub-uniform convergence, which has been given, has assumed the independent variable, x, to be real. This was done for brevity and in

* See chap. XII, § 2.

order that the situation might be more readily visualized. This restriction is in no way essential, however. If x is replaced by the complex variable z the interval, (1), becomes a region, S, of the complex plane; the sub-intervals,

$$p_1, \ldots, p_{m(\epsilon)},$$

become a network of rectangles covering S. The formal wording of the definition and the proof of the fundamental theorem corresponding to Theorem 132 is left to the reader.

§ 4. Infra-uniform convergence.

Definition 26. *If*

$$(1) \qquad \sum_{n=1}^{\infty} u_n(x)$$

converges at all points of

$$(2) \qquad a \leqq x \leqq b,$$

and sub-uniformly over (2) *when and only when any set of intervals covering the points of a certain discrete* set, X, have been removed; then* (1) *is said to converge infra-uniformly over* (2).

We permit X to be empty, that is to contain no points, and thus include sub-uniform convergence as a special case of infra-uniform convergence.

Theorem 133. *Let* (1) *converge at all points of* (2). *Let*

$$\int_a^b u_n(x)\,dx,\ n = 1, 2, \ldots,$$

exist. Let, as usual, $f(x) = \sum_{n=1}^{\infty} u_n(x)$.

A necessary and sufficient condition that $\int_a^b f(x)\,dx$ *exist is that* (1) *converge infra-uniformly over* (2).

All integrals are according to the Riemann definition.

PROOF: $u_n(x)$ is integrable and hence is discontinuous at

* A set of points on a line segment is called discrete, if, given any $\epsilon > 0$, it is possible to set up a finite set of intervals of length $\delta_1, \delta_2, \ldots, \delta_n$, such that each point of the set is on one of these intervals and $\delta_1 + \delta_2 + \ldots + \delta_n < \epsilon$.

a discrete set of points* which, of course, may be empty. An enumerable set of discrete sets of points is also discrete.† Call the set of points arising thus from the discontinuities of the u_n's on (2), Y.

We shall now prove the condition necessary. Assume X to be a non-discrete set. Then $X - Y$ ‡ is also non-discrete. Let P be a point of $X - Y$. Over no interval covering P is $f(x)$ continuous; for, if it were, (1) would converge sub-uniformly over that interval. Any finite set of intervals whatever, including all points of discontinuity of $f(x)$, then includes all points of $X - Y$. Consequently, the points of discontinuity of $f(x)$ form a non-discrete set. Hence, $f(x)$ is not integrable. This is a contradiction; and hence, X must be discrete.

We shall now prove the condition sufficient. If X is discrete, $X + Y$ is also. Hence, the points of discontinuity of $f(x)$ form a discrete set; and hence, $f(x)$ is integrable.

Next, as usual, let $r_n(x) = f(x) - \sum_{n=1}^{\infty} u_n(x)$. Let $\eta_n^{(\delta)}$ be the inferior limit § of the sum of the lengths of a finite number of intervals, including all points where $|r_n(x)| > \delta$.

Theorem 134. HYPOTHESES: (i) (1) *converges infra-uniformly over* (2); (ii) $\sum_{n=1}^{n} |u_n(x)| < M$, *independent of* x *and* n; (iii) $\eta_n^{(\delta)} \to 0$ *for any fixed* δ; (iv) $\int_a^x u_n(x)\,dx$ *exists.*

CONCLUSION:

$$\sum_{n=1}^{\infty} \int_a^x u_n(x)\,dx$$

converges uniformly over (2) *to* $\int_a^x f(x)\,dx$.

PROOF: Let an $\epsilon >$ be given. Take an arbitrary $\delta > 0$.

$$\int_a^x f(x)\,dx = \sum_{n=1}^{n} \int_a^x u_n(x)\,dx + \int_a^x r_n(x)\,dx.$$

* See, for example, Hobson, *Theory of Functions of a Real Vairable*, 1907 ed., p. 342.

† Ibid., p. 105.

‡ Used to denote those points of X not belonging to P.

§ See chap. XII, § 2.

But

$$\left|\int_a^x r_n(x)\,dx\right| \leqq \int_a^x |r_n(x)|\,dx \leqq (x-a)\,\delta + \eta_n^{(\delta)} M$$
$$\leqq (b-a)\,\delta + \eta_n^{(\delta)}\ M < \epsilon,$$

if $\delta < \dfrac{\epsilon}{2(b-a)}$ and n is so great that $\eta_n^{(\delta)} < \dfrac{(b-a)\,\epsilon}{2M}$.

EXERCISES

228. Prove that $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$ converges quasi-uniformly but not uniformly over $-1 \leqq x < 1$.

229. Set up examples of series that are quasi-uniformly but not uniformly convergent and where the interval of convergence is closed.

230. Discuss quasi-uniform convergence of

$$\sum_{n=1}^{\infty} \left[-2\,(n-1)^2 x e^{-(n-1)^2x^2} + 2\,n^2 x e^{-n^2x^2}\right] \text{ over } -1 \leqq x \leqq 1.$$

231. Discuss sub-uniform convergence of

$$\sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right] \text{ over } -1 \leqq x \leqq 1.$$

232. Take each series of exercises 213–218 and examine with reference to quasi- and sub-uniform convergence.

233. Discuss $\sum\limits_{n=1}^{\infty} x^n(1-x)$ and $\sum\limits_{n=0}^{\infty} \dfrac{x^2}{(1+nx^2)(1+(n+1)x^2)}$ especially with reference to infra-uniform convergence.

234. Discuss integrability for each non-uniformly convergent series listed in Exercises 213–218.

235. In the light of this chapter, discuss term by term differentiation of series.

CHAPTER XI

POWER SERIES

§ 1. Fundamental Theorems.

Definition 27. *By the term, power series in z, is meant a series of the form*

$$(1) \qquad c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots,$$

where $c_0, c_1, c_2, c_3, \ldots$, are any numbers whatever but independent of z.

There is no more important class of series. Fundamental theorems are given in this chapter, and the topic is developed at first as if uniform convergence were unknown.

Theorem 135. *The power series* (1) *either* (i) *converges only when $z = 0$, or* (ii) *converges for all values of z, or* (iii) *there exists a fixed positive number, R, such that* (1) *converges absolutely when $|z| < R$ and diverges when $|z| > R$.* In geometric language we say that (1) converges at the origin only or at all points, or at all points inside a circle about the origin and diverges at all points outside this circle.

PROOF: (i) (1) always converges to the value c_0 when $z = 0$. An example of a power series that converges for no value of z other than 0 is $\sum\limits_{n=1}^{\infty} n^n z^n$. This is the case because $n^n z^n \not\rightarrow 0$ when $z \neq 0$. (ii) An example where the series converges for all values of z is $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!} z^n$. This is readily proved by means of Theorem 41. (iii) Suppose (1) converges when $z = a \neq 0$. We shall show that it converges whenever $|z| < |a|$.

$$|c_n z^n| = |c_n a^n| \left|\frac{z}{a}\right|^n.$$

But since (1) converges when $z = a$ there exists a fixed number, M, such that $|c_n a^n| < M$ for all values of n. Hence

$|c_n z^n| < M\left|\frac{z}{a}\right|^n$ ■ the general term of a geometric series with ratio $\left|\frac{z}{a}\right| < 1$. Hence, (1) converges absolutely when $|z| < |a|$.

Next, if (1) diverges when $z = b$ and if $|B| > |b|$ it must diverge when $z = B$; because if it converged when $z = B$ it would necessarily converge when $z = b$, a contradiction.

If $|a| = |b|$ the theorem is proved. If $|b| > |a|$, in geometric language, there exists a circular ring defined by the inequalities $|a| < |z| < |b|$ in which we have ■ yet no knowledge as to the convergence or divergence of (1). Let $|a| = a^{(0)}$ and $|b| = b^{(0)}$. Choose a point, d_1, such that $|d_1| = \frac{1}{2}[a^{(0)} + b^{(0)}]$. The series either diverges or converges when $z = d_1$. In case of convergence replace a by d_1 and denote $|d_1|$ by $a^{(1)}$ and, for symmetry, let $b^{(0)}$ be written $b^{(1)}$. In case of divergence, replace b by d_1 and let $|d_1| = b^{(1)}$ and, for symmetry, let $a^{(0)}$ be written $a^{(1)}$. Now take ■ point d_2, where $|d_2| = \frac{1}{2}[a^{(1)} + b^{(1)}]$, and repeat the process indefinitely. We obtain a sequence of numbers $a^{(0)}, a^{(1)}, \ldots$, such that $a^{(n+1)} \geqq a^{(n)}$, and a sequence $b^{(0)}, b^{(1)}, \ldots$, such that $b^{(n+1)} \leqq b^{(n)}$. Now $b^{(n)} > a^{(0)}$ and $a^{(n)} < b^{(0)}$. Hence, by Theorems 17 and 18, $a^{(n)}$ and $b^{(n)}$ both approach limits. Since

$$b^{(n)} - a^{(n)} = \frac{1}{2^n}[|b| - |a|] \to 0,$$

these limits are equal. Denote this limit by R. It is precisely the R of the theorem and is spoken of as the radius of convergence.

In case (1) converges for all values of z we say that the radius of convergence is infinite $R = \infty$. This is a compact mode of expression and by means of it certain theorems are shortened in statement.

Theorem 135 is the most fundamental theorem with reference to power series.

Theorem 136. *Consider again the power series* (1). HYPOTHESIS: $|c_n a^n|$ *remains finite when* $n \to \infty$. CONCLUSION: (1) *converges whenever* $|z| < |a|$.

PROOF: Suppose $|c_n a^n| \leqq M$ independent of n. Then

$$|c_n z^n| = |c_n a^n|\left|\frac{z}{a}\right|^n \leqq M\left|\frac{z}{a}\right|^n$$

which is the general term of ■ convergent series.

Theorem 137. HYPOTHESIS: *In series* (1) $\left|\frac{c_n}{c_{n+1}}\right| \to R$ *as* $n \to \infty$. CONCLUSION: *R is the radius of the circle of convergence.*

PROOF: This theorem follows immediately from the test ratio test (Theorem 41, Corollary).

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}z^{n+1}}{c_n z^n}\right| = \frac{1}{R}\cdot|z|.$$

The series converges if $\frac{|z|}{R} < 1$, that is if $|z| < R$; and likewise diverges if $|z| > R$.

Theorem 138. *Consider again*

$$(1) \qquad \sum_{n=0}^{\infty} c_n z^n.$$

HYPOTHESIS: *Given any* $\epsilon > 0$, *there exists an* M, *such that when* $n > M$, $\sqrt[n]{|c_n|} < \alpha + \epsilon$, $\alpha > 0$. CONCLUSION: (1) *converges when* $|z| < \frac{1}{\alpha}$.

PROOF: When $n > M$, $\sqrt[n]{|c_n|}\,|z| < (\alpha + \epsilon)|z|$. Consequently, by Theorem 40, (1) converges if $(\alpha+\epsilon)|z| < 1 - \eta$, where $0 < \eta < 1$; that is, if $|z| < \frac{1-\eta}{\alpha+\epsilon}$. By ■ proper choice of η and ■ this can be made as near $\frac{1}{\alpha}$ as we please. Hence, for any value of z, such that $|z| < \frac{1}{\alpha}$, (1) converges.

In line with the theorem just proved is the following theorem. Proof will be omitted.

Theorem 139. HYPOTHESIS: *There exists an M, such that for any $\epsilon > 0$, $\sqrt[n]{|c_n|} > \alpha - \epsilon > 0$, when $n > M$.* CONCLUSION: $\sum_{n=0}^{\infty} c_n z^n$ *diverges when* $|z| > \frac{1}{\alpha}$.

As a corollary to the last two theorems we have:

Theorem 140. HYPOTHESIS: $\sqrt[n]{|c_n|} \to \alpha$. CONCLUSION: $\frac{1}{\alpha}$ *is the radius of the circle of convergence.*

Theorem 141. HYPOTHESIS: $\sum_{n=0}^{\infty} c_n z^n$ *converges when* $|z| < R$. CONCLUSION: $\sum_{n=1}^{\infty} c_n n z^{n-1}$ *converges when* $|z| < R$.

PROOF: Lemma: $1 + 2\theta + 3\theta^2 + 4\theta^3 + \dots$ converges when $|\theta| < 1$. To prove this, apply the test ratio test (Theorem 41, Corollary).

Now to the proof of the theorem.

Give z any particular value such that $|z| < R$. We then can choose a number b such that $|z| < b < R$.

Since $\sum_{n=0}^{\infty} c_n b^n$ converges, there exists a number g, such that $|c_n b^n| < g$ for all values of n. Therefore

$$|nc_n z^{n-1}| = \frac{n}{b}\left|c_n b^n\right|\left|\frac{z}{b}\right|^{n-1} \leqq \frac{g}{b} n \left|\frac{z}{b}\right|^{n-1}.$$

By the lemma, this is the general term of a convergent series. We conclude the theorem.

Theorem 142. HYPOTHESIS: $\sum_{n=1}^{\infty} c_n n z^{n-1}$ *converges when* $|z| < R$. CONCLUSION: $\sum_{n=1}^{\infty} c_n z^n$ *converges when* $|z| < R$ *also.*

PROOF: As before, take any value of z, such that $|z| < R$, and choose b so that $|z| < b < R$. Then, when $n > b$,

$$|c_n z^n| < |c_n n b^{n-1}|.$$

But $\sum_{n=1}^{\infty} |c_n n b^{n-1}|$ converges. The theorem follows by Theorem 36.

Using geometric language, by Theorems 141 and 142, we can say that $\sum_{n=1}^{\infty} c_n z^n$ and $\sum_{n=1}^{\infty} c_n n z^{n-1}$ have the same circle of convergence.

Let $P(z) \equiv c_0 + c_1 z + c_2 z^2 + \dots$ at all points where the series converges.

Theorem 143. *At any point z within the interior of its circle of convergence,*

$$\frac{d}{dz} P(z) = c_1 + 2c_2 z + 3c_3 z^2 + \dots.$$

PROOF: Choose a particular point z_0 inside the circle of convergence and let $h \neq 0$ be so small in absolute value that $|z_0| + |h|$ lies inside the circle also. Then

$$P(z_0 + h) - P(z_0) = \sum_{n=0}^{\infty} c_n [(z_0 + h)^n - z_0^n]$$

$$= \sum_{n=1}^{\infty} c_n [n z_0^{n-1} h + \dots + h^n]$$

converges absolutely.

$$\frac{P(z_0 + h) - P(z_0)}{h} = \sum_{n=1}^{\infty} c_n \left[n z_0^{n-1} + \frac{n(n-1)}{2!} z_0^{n-2} h + \dots + h^{n-1} \right].$$

Let $A(z) = \sum_{n=1}^{\infty} c_n n z^{n-1}$. Then

$$\frac{P(z_0 + h) - P(z_0)}{h} - A(z_0) = \sum_{n=1}^{\infty} c_n \left[\frac{n(n-1)}{2!} z_0^{n-2} h + \dots + h^{n-1} \right].$$

But

$$\left| c_n \left[\frac{n(n-1)}{2!} z_0^{n-2} h + \dots + h^{n-1} \right] \right|$$

$$\leqq |h| |c_n| \left[\frac{n(n-1)}{2!} |z_0|^{n-2} + \dots + |h|^{n-2} \right].$$

And $$\sum_{n=2}^{\infty} |c_n| \left[\frac{n(n-1)}{2!}|z_0|^{n-2} + \ldots + |h|^{n-2}\right]$$
converges; for
$$\frac{n(n-1)}{2!}|z_0|^{n-2} + \ldots + |h|^{n-2} \leqq \frac{(|z_0|+|h|)^n}{h^2},$$
and $$\sum_{n=1}^{\infty} c_n(|z_0|+|h|)^n$$
converges. Consequently
$$\left|\frac{P(z_0+h)-P(z_0)}{h} - A(z_0)\right|$$
$$\leqq |h| \sum_{n=2}^{\infty} |c_n| \left[\frac{n(n-1)}{2!}|z_0|^{n-2} + \ldots + |h|^{n-2}\right].$$
Call this last expression $|h|B(h)$. When $|h|$ decreases each term of $B(h)$ but the first decreases; and, as $B(h)$ is a series of terms $\geqq 0$, $B(h)$ does not increase. Since
$$\left|\frac{P(z_0+h)-P(z_0)}{h} - A(z_0)\right| \leqq hB(h),$$
$$\lim_{h\to 0} \frac{P(z_0+h)-P(z_0)}{h} = A(z_0),$$
which establishes the theorem.

Corollary 1. *Since $A(z)$ has the same circle of convergence as $P(z)$,*
$$\frac{d^n}{dz^n}P(z) = c_n \cdot n! + c_{n+1}\frac{(n+1)!}{1!}z + c_{n+2}\frac{(n+2)!}{2!}z^2 + \ldots.$$

Corollary 2. *$P(z)$, as well as each of its derivatives, defines a continuous function over the interior of its circle of convergence.*

Theorem 144. HYPOTHESIS: *Given any $\delta > 0$ whatsoever, $P(z) \equiv a_0 + a_1 z + a_2 z^2 + \ldots$ vanishes for a value of z other than $z = 0$ when $|z| < \delta$.* CONCLUSION:
$$a_0 = a_1 = \ldots = a_n = \ldots = 0.$$

PROOF: Suppose $P(z_0) = 0$. Then $P(z)$ is continuous when $|z| < |z_0|$, by Theorem 135 and Corollary 2 to Theorem 143.

Consequently, by the definition of continuity, given any $\epsilon > 0$ we can choose a δ such that $|P(0)-P(z)| < \epsilon$ when $|z| < \delta$. But $P(z)$ is zero for certain values of z in absolute value less than δ. Hence $|P(0)-0| < \epsilon$; and, consequently, $P(0) = 0$. But $P(0) = a_0$.

Now suppose $a_0 = \ldots = a_{n-1} = 0$; to show $a_n = 0$.
$$P(z) = z^n(a_n + a_{n+1}z + a_{n+2}z^2 + \ldots).$$
At those points where $P(z) = 0$, other than $z = 0$,
$$P_n(z) \equiv a_n + a_{n+1}z + a_{n+2}z^2 + \ldots$$
vanishes. It is a continuous function when $|z| < |z_0|$; and moreover $P_n(0) = a_n$. As before, $|a_n - 0| < \epsilon$ when $|z| < \delta$, and hence $a_n = 0$.

Induction from the fact that $a_0 = 0$ gives the conclusion of the theorem.

Corollary. HYPOTHESIS: *No matter how small a $\delta > 0$ is taken,*
$$a_0 + a_1 z + a_2 z^2 + \ldots = b_0 + b_1 z + b_2 z^2 + \ldots$$
for some value of z, other than 0, but in absolute value less than δ. CONCLUSION: $b_n = a_n$, $n = 0, 1, 2, \ldots$.

Theorem 145. HYPOTHESIS: *$F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges when $|z| < M$.* CONCLUSION: *Let c be any curve lying wholly within the circle $|z| = M$ and of length $< A$. Then*
$$\int_c F(z)\,dz = \sum_{n=0}^{\infty} c_n \int_c z^n\,dz.$$

PROOF: Let a and z be the limits of integration on c. Then $$\int_c z^n\,dz = \frac{1}{n+1}(z^{n+1} - a^{n+1}).$$
By Theorems 135 and 142,
$$\sum_{n=0}^{\infty} \frac{c_n}{n+1}(z^{n+1} - a^{n+1})$$
converges within the circle $|z| = M$. Denote the value of its sum by $\phi(z)$. By Theorem 143 we have
$$\frac{d}{dz}\phi(z) = F(z).$$

That is
$$\phi(z) = \int F(z)\,dz + K,$$
where K is a constant. Letting $z = a$ we find $K = 0$, that is
$$\phi(z) = \int_0 F(z)\,dz,$$
which is what we wished to prove.

Theorem 146. HYPOTHESES: (i) $\sum_{n=0} c_n R^n$ *converges to* S; (ii) $0 < \theta < 1$. CONCLUSION: $\lim_{\theta \to 1} \sum_{n=0}^{\infty} c_n \theta^n R^n = S.$

PROOF:
$$\left| S - \sum_{n=0}^{\infty} c_n (\theta R)^n \right|$$
$$\leqq |c_1|.|R|.|1-\theta| + \dots + |c_M|.|R^M|.|1-\theta^M|$$
$$+ |c_{M+1}R^{M+1} + \dots + c_{M+p}R^{M+p} + \dots|$$
$$+ |c_{M+1}\theta^{M+1}R^{M+1} + \dots + c_{M+p}\theta^{M+p}R^{M+p} + \dots|.$$

Now let an $\epsilon > 0$ be given. Due to the convergence of the series given in hypothesis (i), if M is taken large enough,
$$|c_{M+1}R^{M+1} + \dots + c_{M+p}R^{M+p}| < \frac{\epsilon}{3}$$
for all values of $p \geqq 0$. Moreover, the expression in the second line of the right-hand member above, being the limit of this as $p \to \infty$, will be less than or equal to $\frac{\epsilon}{3}$. Now, by Theorem 28,
$$|c_{M+1}\theta^{M+1}R^{M+1} + \dots + c_{M+p}\theta^{M+p}R^{M+p}| < \theta^{M+1}\frac{\epsilon}{3}.$$
Consequently, the expression in the third line of the right-hand member is less than $\frac{\epsilon}{3}$. Next choose a number G such that $|c_n R^n| < G$ for all values of n. Then choose θ so near 1 that $|1-\theta^n| < \frac{\epsilon}{3\,MG}$ when $n \leqq M$. When this is the case, the expression in the first line is less than $\frac{\epsilon}{3}$. It results that
$$\left| S - \sum_{n=0}^{\infty} c_n (\theta R)^n \right| < \epsilon,$$
which establishes the theorem.

We proceed to a generalization of the preceding theorem. The only reason for giving that theorem at all is that it is in substantially that form that the theorem is generally known.

Theorem 147. HYPOTHESES: (i) $\sum_{n=1}^{\infty} c_n R^n$ *converges to* S: (ii) $0 < |\theta| < 1$; (iii) $\frac{|1-\theta|}{1-|\theta|} < K$, *a constant.* CONCLUSION:
$$\lim_{\theta \to 1} \sum_{n=0}^{\infty} c_n \theta^n R^n = S.$$

Here θ may be complex, and there is no restriction on its manner of approach to unity, except that as a point in the complex plane it remain within some fixed angle formed by two half-lines meeting at the point 1 and inclined from the vertical toward the origin.

PROOF: Let an $\epsilon > 0$ be given.
$$\left| S - \sum_{n=0}^{\infty} c_n \theta^n R^n \right| \leqq |c_1(1-\theta)R + c_2(1-\theta^2)R^2 + \dots$$
$$+ c_M(1-\theta^M)R^M| + |c_{M+1}R^{M+1} + c_{M+2}R^{M+2} + \dots|$$
$$+ |c_{M+1}\theta^{M+1}R^{M+1} + c_{M+2}\theta^{M+2}R^{M+2} + \dots|.$$
Choose M so that
$$|c_{M+1}R^{M+1} + c_{M+2}R^{M+2} + \dots + c_{M+p}R^{M+p}| < \delta$$
for all values of $p \geqq 1$. Then
$$|c_{M+1}R^{M+1} + c_{M+2}R^{M+2} + \dots| \leqq \delta.$$
Now choose θ so close to 1 that for it and all closer values
$$|c_1(1-\theta)R + c_2(1-\theta^2)R^2 + \dots + c_M(1-\theta^M)R^M| < \delta.$$

Now apply Theorem 86 to

$$c_{M+1}\theta^{M+1}R^{M+1}+c_{M+2}\theta^{M+2}R^{M+2}+\dots,$$

letting $b_0=\theta^{M+1}$, $b_1=\theta^{M+2}$, ...

$$|c_{M+1}\theta^{M+1}R^{M+1}+c_{M+2}\theta^{M+2}R^{M+2}+\dots|$$

$$\leqq \delta\sum_{n=1}^{\infty}|\theta^{M+n-1}-\theta^{M+n}|=\delta|\theta|^{M}\sum_{n=1}^{\infty}|\theta^{n-1}-\theta^{n}|$$

$$=\delta|\theta|^{M}\frac{|1-\theta|}{1-|\theta|}<\delta K.$$

Then $$|S-\sum_{n=0}^{\infty}c_n\theta^nR^n|<\delta(2+K)<\epsilon$$

if $\delta<\dfrac{\epsilon}{2+K}$, proving the theorem.

We insert next a theorem which is a departure from our more recent theorems as it brings in a second variable. The theorem, however, is on the general topic of continuity and consequently is introduced at this point.

Theorem 148. HYPOTHESES: (i) *$f_n(y), n=1,2,\dots,$ is continuous over a region, S;* (ii) *there exist positive constants, x_0, A, and p, such that for all values of n,*

$$|f_n(y)|\,x_0^n<An^p;$$

(iii) $|z|<x_0$. CONCLUSION: $\sum\limits_{n=1}^{\infty}f_n(y)z^n$ *converges to a function of y and z which is continuous in y over S.*

PROOF: $|f_n(y)z^n|<An^p\left(\dfrac{|z|}{x_0}\right)^n$ which is the general term of a convergent series. The theorem follows by Theorem 113, corollary, coupled with Theorem 111.

Compare Theorem 148 with Theorem 105.

§ 2. Dominant function.

Definition 28. *Given $f(z)=a_0+a_1z+a_2z^2+\dots$ and $\phi(z)=b_0+b_1z+b_2z^2+\dots$, both convergent when $|z|<R$ and such that $b_n\geqq 0$ and $|a_n|\leqq b_n$; then $\phi(z)$ is said to dominate $f(z)$ over their common region of convergence.*

The notation $f(z)\ll\phi(z)$ will be employed.

It is to be observed that within their circles of convergence, power series obey the same rules of operation and generally behave as do polynomials. Moreover, if $P(x_1,x_2,\dots,x_n)$ is a polynomial with positive coefficients,

$$|P(a_0,a_1,\dots a_{n-1})|\leqq P(b_0,b_1,\dots,b_{n-1}).$$

Consequently, if $f(z)\ll\phi(z)$, $[f(z)]^2\ll[\phi(z)]^2$, &c.

Theorem 149. HYPOTHESES: (i) *$f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ converges when $|z|<R$;* (ii) *$|z|<r<R$;* (iii) *M is a number such that $|a_nr^n|\leqq M$ for every value of n.* CONCLUSION:

$$f(z)\ll M+M\frac{z}{r}+\dots+M\frac{z^n}{r^n}+\dots=\frac{M}{1-\frac{z}{r}}.$$

PROOF: $|a_nr^n|\leqq M$ by hypothesis. Consequently $|a_n|\leqq\dfrac{M}{r^n}$, which is the required proof.

Other dominant functions might be set up, as dominance is in no way a unique property. For example, if $a_0=0$,

$$f(z)\ll M\frac{z}{r}+\dots+M\frac{z^n}{r^n}=\frac{M}{1-\frac{z}{r}}-M.$$

Theorem 150. *Use the notation and series of the previous theorem.* HYPOTHESIS: $a_0\neq 0$. CONCLUSION: *It is possible to find a positive $\rho<R$, such that*

$$f(z)\ll\frac{|a_0|}{1-\frac{z}{\rho}}.$$

PROOF: Let a particular dominant function, as derived in the previous theorem, be $M+M\dfrac{z}{r}+M\dfrac{z^2}{r^2}+\dots$. Choose $0<\rho<r\dfrac{|a_0|}{M}$. Then

$$|a_n\rho^n|=|a_nr^n|\cdot\left(\frac{\rho}{r}\right)^n<M\frac{\rho}{r}\left(\frac{\rho}{r}\right)^{n-1}.$$

Now, if necessary, take ρ, still greater than zero, so small that

$$M\frac{\rho}{r} \leqq |a_0|.$$

Then $$|a_n \rho^n| < |a_0| \quad \text{or} \quad |a_n| < \frac{|a_0|}{\rho^n}.$$

Hence $$f(z) \ll |a_0| + |a_0|\frac{z}{\rho} + |a_0|\frac{z^2}{\rho^2} + \dots = \frac{|a_0|}{1-\dfrac{z}{\rho}}.$$

Dominant functions are useful in many places in mathematics. We have in no way exhausted the subject. This section is introduced simply to familiarize the reader with the idea.

§ 3. Substitution of one power series in another.

Theorem 151. HYPOTHESES: (i) $\phi(y) = b_0 + b_1 y + b_2 y^2 + \dots$ *converges when* $|y| < S$; (ii) $y = a_0 + a_1 z + a_2 z^2 + \dots$ *converges when* $|z| < R$; (iii) $|a_0| < S$; (iv) h *is any particular positive number such that when* $|z| < h$, $\sum_{n=0}^{\infty} |a_n| . |z^n| < S$. CONCLUSION: *when* $|z| < h$, $\phi(y) = \sum_{n=0}^{\infty} c_n z^n$,

where
$$\begin{aligned} c_0 &= b_0 + b_1 a_0 + b_2 A_0^{(1)} + b_3 A_0^{(2)} + \dots \\ c_1 &= b_1 a_1 + b_2 A_1^{(1)} + b_3 A_1^{(2)} + \dots \\ c_2 &= b_1 a_2 + b_2 A_2^{(1)} + b_3 A_2^{(2)} + \dots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

The $A_i^{(j)}$*'s are obtained by squaring, cubing, &c., the series* $a_0 + a_1 z + a_2 z^2 + \dots$ *and indicating the resulting series by*

$$\begin{gathered} A_0^{(1)} + A_1^{(1)} z + A_2^{(1)} z^2 + \dots \\ A_0^{(2)} + A_1^{(2)} z + A_2^{(2)} z^2 + \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{gathered}$$

PROOF: Consider the double series obtained by substituting for y in the series $\phi(y)$, namely

(1) $$\begin{aligned} & b_0 + 0 + 0 + \dots \\ &+ b_1 a_0 + b_1 a_1 z + b_1 a_2 z^2 + \dots \\ &+ b_2 A_0^{(1)} + b_2 A_1^{(1)} z + b_2 A_2^{(1)} z^2 + \dots \\ &+ b_3 A_0^{(2)} + b_3 A_1^{(2)} z + b_3 A_2^{(2)} z^2 + \dots \\ &+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

If we sum by columns we get the desired result. We can do this if the double series converges. To prove convergence, consider the double series of absolute values. Its first row is $|b_0|$, its second row is $|b_1|\bar{Y}$, where

$$\bar{Y} = |a_0| + |a_1||z| + |a_2||z^2| + \dots.$$

The third row is dominated by $|b_2|\bar{Y}^2$, and in general the $(n+1)$th row is dominated by $|b_n|\bar{Y}^n$. Consequently, when $\bar{Y} < S$, that is when $|z| < h$, the double series of absolute values converges. This means that the double series itself converges.

Theorem 152. HYPOTHESES: (i) $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$, *converges when* $|z| < R_1$; (ii) $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ *converges when* $|z| < R_2$. CONCLUSION: $\sum_{n=0}^{\infty} c_n z^n$, *where* $c_0, c_1, \dots$ *are determined by dividing* $\phi(z)$ *by* $\psi(z)$ *by the usual rule for dividing polynomials arranged according to ascending powers of* z, *converges to* $\dfrac{\phi(z)}{\psi(z)}$ *when* $|z| < R_1$, $|z| < R_2$, *and when in addition* $|z|$ *is so small that* $|b_1 z + b_2 z^2 + \dots| < |b_0|$.

PROOF: Let $y = \dfrac{1}{b_0}(b_1 z + b_2 z^2 + \dots)$. Consider

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \dots.$$

This converges if $|y| < 1$, which we have assumed. Replacing y by its value according to Theorem 151, we have

$$\frac{1}{b_0}\frac{1}{1+y} = \frac{1}{b_0} - \frac{b_1}{b_0^2}z + \frac{b_1^2 - b_2 b_0}{b_0^3}z^2 + \dots$$

which is convergent when $|z| < R_1$. In addition let $|z| < R_2$ and multiply the series just obtained by $\sum_{n=0}^{\infty} a_n z^n$. We obtain the desired convergent series, $\sum_{n=0}^{\infty} c_n z^n$. See Theorem 79.

The coefficients are those which would be obtained by the usual division process as we proceed to show. The process employed here applied to the polynomials $\sum_{n=0}^{n} a_n z^n$ and $\sum_{n=0}^{n} b_n z^n$ yields the coefficients, $c_0, c_1, \dots c_n$. These coefficients are also obtained by the usual division process. In both processes the work is identical with that for finding the first n coefficients in the infinite series.

Theorem 153. HYPOTHESES: *These are the same as in the previous theorem except that* $b_0 = \dots = b_{k-2} = 0$, $b_{k-1} \neq 0$. CONCLUSION: *Divide* $\phi(z)$ *by* $\psi(z)$, *formally obtaining*

$$\frac{c_0}{z^{k-1}} + \frac{c_1}{z^{k-2}} + \dots + \frac{c_{k-2}}{z} + c_{k-1} + c_k z + \dots .$$

$\sum_{n=k-1}^{\infty} c_n z^{n-k+1}$ *converges when* $|z| < R_1$, $|z| < R_2$, *and in addition* $|z|$ *is so small that*

$$|b_k z + b_{k+1} z^2 + \dots| < b_{k-1}.$$

Moreover, if $z \neq 0$,

$$\frac{\phi(z)}{\psi(z)} = \frac{c_0}{z^{k-1}} + \frac{c_1}{z^{k-2}} + \dots + \frac{c_{k-1}}{z} + c_{k-1} + c_k z + \dots .$$

PROOF: $\psi(z) = z^{k-1}[b_{k-1} + b_k z + \dots]$. Let

$$\chi(z) = [b_{k-1} + b_k z + \dots]$$

and develop $\frac{\phi(z)}{\chi(z)}$ according to the previous theorem. Multiply by $\frac{1}{z^{k-1}}$ and the desired development is obtained.

§ 4. Uniform convergence of power series.

Theorem 154.* HYPOTHESES: (i) $\sum_{n=0}^{\infty} c_n z^n$ *converges when* $z = z_0$; (ii) $0 < R < |z_0|$, *where* R *is a constant.* CONCLUSION: $\sum_{n=0}^{\infty} c_n z^n$ *converges uniformly over the region defined by the inequality,* $|z| \leq R$.

* Cf. Theorem 135.

PROOF: The series $\sum_{n=0}^{\infty} |c_n| R^n$ serves ■ the comparison series of Theorem 113 and proof is immediate.

Theorem 155.* HYPOTHESIS: $\sum_{n=0}^{\infty} a_n x^n$ *converges when* $x = 1$. CONCLUSION: *It converges uniformly over the interval*

$$0 \leq x \leq 1.$$

PROOF: Let $r_m(x) = \sum_{n=m}^{\infty} a_n x^n$ and let

$$s_{m,n} = a_m + a_{m+1} + \dots + a_{m+n-1}.$$

Let ■ > ■ be given.

When $|x| < 1$,

$$r_m(x)\frac{1}{1-x} = r_m(x) \cdot (1 + x + x^2 + \dots) = \sum_{n=1} s_{m,n} x^{m+n-1}.$$

Now take M so large, that when $m > M$, $|s_{m,n}| < \epsilon$ for every $n \geqq 1$, then

$$|r_m(x)| = \left|(1-x)\sum_{n=1}^{\infty} s_{m,n} x^{m+n-1}\right|$$

$$< |1-x| \cdot \epsilon \cdot \left|\sum_{n=1}^{\infty} x^{m+n-1}\right| < (1-x) \cdot \epsilon \cdot \sum_{n=1}^{\infty} x^n = \epsilon.$$

This is when $0 \leqq x < 1$. Moreover, when $x = 1$,

$$r_m = \lim_{n \to \infty} s_{m,n},$$

and consequently $|r_m(1)| \leqq \epsilon$ when $m > M$. Hence

$$|r_m(x)| \leqq \epsilon,$$

when $m > M$ and $0 \leqq x \leqq 1$; which establishes uniform convergence.

Corollary 1. *The point* 1 *can be replaced by any point,* x_0, *on the real axis.*

For proof replace $a_n x_0^n$ by a_n and x by $\frac{x}{x_0}$ and the corollary reduces to the theorem.

* Cf. Theorem 146.

Corollary 2. *The interval on the X-axis can be replaced by an interval on a ray from the origin to a point z_0; that is, x is replaced by $xe^{\phi i}$.*

For proof, amalgamate the $e^{n\phi i}$ with a_n. This corollary then reduces to Corollary 1.

The last theorem is a corollary to the following more general theorem. It is given independently on account of the great importance of the topic and the simplicity of its proof. The following theorem, although a ready consequence of Theorems 147 and 121, is essentially more difficult than 155.

Theorem 156. HYPOTHESIS: $\sum_{n=0}^{\infty} c_n z^n$ *converges when* $z = z_0$.

CONCLUSION: *It converges uniformly over any region in the complex plane bounded by two straight lines meeting at z_0 and extending within the circle about the origin through z_0, and any other curve lying wholly within this circle.*

Details of proof are omitted. See Theorems 147 and 121.

From the last several theorems results as to continuity, integrability, differentiability, &c., of power series can readily be drawn. See Theorems 111, 122, 123, 124, also Theorems 143 and 145, where independent proofs are given.

§ 4. Inversion * of power series.

Theorem 157. HYPOTHESIS:

$$(1) \qquad u = \sum_{n=0}^{\infty} b_n z^n,\ b_1 \neq 0,$$

converges when $|z| \leqq |z_0|$, $|z_0| > 0$. CONCLUSION: *There exists an $h > 0$, such that when $|u - b_0| < h$ there is a unique series of the type*

$$(2) \qquad z = (u - b_0)\left\{\frac{1}{b_1} + a_1(u-b_0) + a_2(u-b_0)^2 + \ldots\right\},$$

which converges and satisfies (1).

* Sometimes called 'Reversion'.

PROOF: Replace z by tz_0 and let $v = \dfrac{(u-b_0)}{b_1 z_0}$. Series (1) reduces to a series of the form

$$(3) \qquad v = t - c_2 t^2 - c_3 t^3 - \ldots,$$

which converges when $|t| \leqq 1$. Series (2) reduces to a series of the form

$$(4) \qquad t = v(1 + d_1 v + d_2 v^2 + \ldots).$$

These series are the original only written in a different way. In other words, if t is given by the latter series the corresponding value of z is given by the former, &c. There consequently is no loss of generality in treating the latter series. We proceed to do so.

Substitute (4) in (3), assuming for the moment the convergence of (4). We get

$$\begin{aligned} v = v &+ (d_2 - c_2)v^2 + (d_3 - 2c_2 d_2 - c_3)v^3 \\ &+ [d_4 - c_2(d_2^2 + 2d_3) - 3c_3 d_2 - c_4]v^4 \\ &+ [d_5 - 2c_2(d_4 + d_2 d_3) - 3c_3(d_2^2 + d_3) - 4c_4 d_2 - c_5]v^5 + \ldots. \end{aligned}$$

Equate coefficients of like powers of v, by Theorem 144, and we have

$$\begin{aligned} d_2 &= c_2, \\ d_3 &= 2c_2 d_2 + c_3, \\ d_4 &= c_2(d_2^2 + 2d_3) + 3c_3 d_2 + c_4, \\ d_5 &= 2c_2(d_4 + d_2 d_3) + 3c_3(d_2^2 + d_3) + 4c_4 d_2 + c_5, \\ &\ldots \\ d_n &= f_n(c_1, \ldots, c_{n-1}) + c_n, \end{aligned}$$

f_n being a polynomial with positive coefficients. We see that the c_n's are always uniquely determinable. This determination gives a formal solution of the problem. It remains, then, only to prove the convergence of (4) for values other than 0.

(3) converges when $t = 1$. Hence, $|c_n| < \alpha$, a constant; and

$$(5) \qquad |d_n| \leqq f_n(\alpha, \ldots, \alpha) + \alpha = F_n(\alpha, \ldots, \alpha).$$

Now let us consider

$$(6) \qquad v = t - \alpha t^2 - \alpha t^3 - \ldots = t - \frac{\alpha t^2}{1-t}.$$

Solving for t we have

(7) $$t = \frac{1+v \pm \sqrt{1-2(2\alpha+1)v+v^2}}{2(1+\alpha)}.$$

Consider that the radical is so determined in the neighbourhood of $v = 0$ that, when $v = 0$, it equals 1. Write it as

$$(1-2(2\alpha+1)v+v^2)^{\frac{1}{2}},$$

and expand by the binomial theorem, getting a series that converges when $|2(2\alpha+1)v-v^2| < 1$ and which can be written as a power series in v according to Theorem 151. Substitute this in (7). The series obtained, where the minus sign is used before the radical, coincides with that got by the method of undetermined coefficients as applied to (6). Because both satisfy (6), and we have seen in the discussion of (3) that when a power series for t in terms of v, with constant term zero, satisfies (6), its coefficients are uniquely determined. Its $(n+1)$st coefficient is $F_n(\alpha, \ldots, \alpha)$. And, since $|d_n| < F_n(\alpha, \ldots, \alpha)$, (4) converges and the theorem is complete.

§ 5. Power series in several variables.

Definition 29. *A series of the type*

(1) $$\sum_{n=0,\, m=0}^{\infty} a_{n,m} x^m y^n,$$

where the $a_{n,m}$'s are constants, is called a double power series in x and y.

The following theorem is a generalization of Theorem 154.

Theorem 158. HYPOTHESES: (i) *$x_0 \neq 0$ and $y_0 \neq 0$ are fixed and M is a positive number*; (ii)

$$|a_{n,m} x_0^m y_0^n| < M$$

for all values of n and m. CONCLUSION: (1) *converges uniformly* when $|x| \leqq X < |x_0|$ and $|y| \leqq Y < |y_0|$ simultaneously, X and Y being fixed.*

* The notion of uniform convergence for multiple series is assumed without formal definition.

Proof is made by comparison with the double series of positive terms $\sum_{n=0,\, m=0}^{\infty} M\left|\frac{x}{x_0}\right|^m \cdot \left|\frac{y}{y_0}\right|^n$ which converges to

$$\frac{M}{\left(1-\left|\frac{x}{x_0}\right|\right)\left(1-\left|\frac{x}{y_0}\right|\right)}.$$ See Theorem 91.

The following theorem also can be proved readily. Incidentally see Theorem 135.

Theorem 159. HYPOTHESIS: (1) *diverges when $x = x_0$ and $y = y_0$ simultaneously.* CONCLUSION: (1) *diverges when $|x| \geqq |x_0|$ and $|y| \geqq |y_0|$ simultaneously.*

If we consider the double series (1) we can differentiate or integrate term by term as many times as desired. We infer this from the uniform convergence. An exact statement and independent proof, however, is given.

Theorem 160. HYPOTHESIS:

(1) $$\sum_{n=0,\, m=0}^{\infty} a_{m,n} x^n y^m = F(x, y)$$

converges when $|x| < a$ and $|y| < b$ simultaneously. CONCLUSION:

(2) $$\sum_{n=1,\, m=0}^{\infty} a_{m,n} n x^{n-1} y^m$$

converges to $\dfrac{\partial F(x, y)}{\partial x}$ *and*

(3) $$\sum_{n=0,\, m=1}^{\infty} a_{m,n} m x^n y^{m-1}$$

to $\dfrac{\partial F(x, y)}{\partial y}$ *whenever $|x| < a$ and $|y| < b$ simultaneously.*

PROOF: Consider (2) for example, and suppose first that $a_{m,n}$, x, and y are all positive reals. Then, by Theorem 90,

$$F(x, y) = \sum_{n=0} x^n \sum_{m=0}^{\infty} a_{m,n} y^m.$$

Also, since (2) is now a series of positive terms, it converges if

$$\sum_{n=0}^{\infty} nx^{n-1} \sum_{m=0}^{\infty} a_{m,n} y^m$$

converges and to the same value. But, by Theorem 143,

$$\sum_{n=0}^{\infty} nx^{n-1} \sum_{n=0}^{\infty} a_{m,n} y^m$$

converges to

$$\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} a_{m,n} y^m\right] = \frac{\partial F(x,y)}{\partial x}.$$

Now suppose the $a_{m,n}$'s, x, and y any complex numbers. Consider the double series formed of the absolute values of the terms of (2). It converges, as we know for the following reasons. (1) converges, that is it converges absolutely. Differentiate this series of absolute values and we get a convergent double series by the proof that we have just gone through with. It is the series of absolute values of (2) of which we spoke. Hence, (2) converges and can be summed by columns or rows (see Theorem 90). Consequently

$$\sum_{n=1, m=0}^{\infty} a_{m,n} nx^{n-1} y^m = \sum_{n=1}^{\infty} nx^{n-1} \sum_{m=0}^{\infty} a_{m,n} y^m$$
$$= \frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} a_{m,n} y^m\right] = \frac{\partial F(x,y)}{\partial x}$$

by Theorem 143.

The proof of the formula for finding $\frac{\partial F(x,y)}{\partial y}$ requires only obvious modifications of that given.

Corollary. *Series (2) and (3) are of the form (1); and consequently (1) can be differentiated term by term as many times as desired.*

Theorem 161. HYPOTHESES: (i)

$$\sum_{n=0, m=0}^{\infty} a_{m,n} x^n y^m = F(x,y)$$

converges when $|x| < a$, $|y| < b$; (ii) *c is a curve lying wholly within the circle* $|y| = b$; (iii) γ *is a curve lying wholly within the circle* $|x| = a$. CONCLUSIONS: (i) *When* $|x| < a$.

$$\int_c F(x,y)\,dy = \sum_{n=0, m=0}^{\infty} a_{m,n} x^n \int_c y^m\,dy;$$

(ii) *when* $|y| < b$, $\displaystyle\int_\gamma F(x,y)\,dx = \sum_{n=0, m=0}^{\infty} a_{m,n} y^m \int_\gamma x^n\,dx.$

PROOF: By Theorem 90,

$$\sum_{n=0, m=0}^{\infty} a_{m,n} x^n y^m = \sum_{m=0}^{\infty}\left[y^m \sum_{n=0}^{\infty} a_{m,n} x^n\right];$$

and hence

$$\int_c F(x,y)\,dy = \sum_{m=0}^{\infty}\left[\left(\int_c y^m\,dy\right) \sum_{n=0}^{\infty} a_{m,n} x^n\right].$$

By Theorem 90, this equals

$$\sum_{n=0, m=0}^{\infty} a_{m,n} x^n \int_c y^m\,dy.$$

provided the latter series converges. To prove this convergence, of course holding $|x| < a$, let g be the lower limit of integration and y the upper. Then

$$\sum_{n=0, m=0}^{\infty} a_{m,n} x^n \int_c y^m\,dy$$
$$= \sum_{n=0, m=0}^{\infty} \frac{a_{m,n}}{m+1} x^n y^{m+1} - \sum_{n=0, m=0}^{\infty} \frac{a_{m,n}}{m+1} x^n g^{m+1}.$$

Each of these series in the right-hand member converges by comparison with the absolute value series of (1); because, when $m+1 > |y|$ and $> |g|$, $\frac{1}{m+1}|y^{m+1}| < |y^m|$ and $\frac{1}{m+1}|g^{m+1}| < |g^m|$. See Theorem 91.

This completes the proof, so far as the first half of the conclusion is concerned. The second half can be proved in the same manner.

In the next theorem we treat the question of substitution in double power series.

Theorem 162. HYPOTHESIS:

(1) $$F(x, y) = \sum_{m=0,\, n=0}^{\infty} a_{m,n} x^m y^n$$

converges when $|x| < a$ *and* $|y| < b$ *simultaneously,*

(4) $$x = a_0 + a_1 z + a_2 z^2 + \ldots, \text{ and}$$

(5) $$y = b_0 + b_1 z + b_2 z^2 + \ldots,$$

where $|a_0| < a$ *and* $|b_0| < b$, *converge when* $|z| < c$. CONCLUSION: *There exists a fixed* $h > 0$ *such that when* $|z| < h$, $F(x, y)$ *is given by a series*

(6) $$A_0 + A_1 z + A_2 z^2 + \ldots,$$

where the coefficients $A_0, A_1, A_2, \ldots$ *are obtained by substituting from* (4) *and* (5) *in* (1), *collecting those terms containing the same power of* z *and summing the resulting infinite series.*

PROOF: Arrange the double series (1) in a horizontal (m, n) plane. Power series can be multiplied within their interval of convergence. Substitute for x and y the series (4) and (5) and arrange the resulting power series in z ▪ vertical columns, like powers of z being in the same horizontal plane. The triple series thus obtained, which we call (7), converges. For consider the series of absolute values. Suppose that when $z = z_0$, $0 < z_0 < c$, the series

(8) $$|a_0| + |a_1||z| + |a_2||z|^2 + \ldots \quad \text{and}$$

(9) $$|b_0| + |b_1||z| + |b_2||z|^2 + \ldots$$

converge to values less than a and b respectively. We know that it is possible to find such ▪ 'z_0', since (4) and (5) converge absolutely when $|z| < c$ and since power series such as (8) and (9) represent continuous functions, and $|a_0| < a$ and $|b_0| < b$. The absolute value series, when $|z| < z_0$, has its columns converge and the double series of values converges. Consequently it converges. The fact that the absolute value series converges implies the convergence of (7). Now since (7) converges, it can be summed by planes parallel to the (m, n) plane, by Theorem 96. This gives us the desired series (6), of the theorem.

The idea of Dominant Function for power series in several variables is a generalization of that under Section 2.

Definition 30. *Given a double power series,* $f(x, y)$; *we say that a second double power series in the same two variables,* $\phi(x, y)$, *dominates* $f(x, y)$ *over their common region of convergence, if each coefficient of* $\phi(x, y)$ *is positive and as great as the absolute value of the corresponding coefficient of* $f(x, y)$.

If $\sum_{n=0,\, m=0}^{\infty} a_{m,n} x^m y^n$ converges when $|x| < a$ and $|y| < b$ and $|a_{m,n}| < M$ for all values of m and n, it is readily shown, as in Section 2, that

$$\phi(x, y) = M \sum_{n=0,\, m=0}^{\infty} \frac{x^m}{a^m} \cdot \frac{y^n}{b^n} = \frac{M}{\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)}$$

dominates $\sum_{n=0,\, m=0}^{\infty} a_{m,n} x^m y^n$. Another dominant series is the double series expansion of

$$\psi(x, y) = \frac{M}{1 - \left(\frac{x}{a} + \frac{y}{b}\right)}.$$

For the coefficient of $x^m y^n$ in $\psi(x, y)$ is equal to the coefficient of the corresponding term in the expansion of

$$M\left(\frac{x}{a} + \frac{y}{b}\right)^{m+n},$$

and hence is at least equal to $\frac{M}{a^m b^n}$ the coefficient of $x^m y^n$ in $\phi(x, y)$.

The definitions and theorems of this section have been given for double power series. This has been primarily for brevity in statement. And the reader should have no difficulty in generalizing to a larger number of variables.

EXERCISES

236–246. Determine the circle of convergence of the power series given below. Examine for convergence at points on the circle of convergence whenever it is finite. [HINT: In considering these series for complex values of z on the circle of convergence Theorems 65, 85, and 86 will be especially useful. The problem is frequently difficult.]

$$z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots;$$

$$z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \dots;$$

$$1 - z^2 - \frac{z^4}{2!} - \frac{z^6}{3!} + \frac{z^8}{4!} - \dots;$$

$$z - \left(1 + \frac{1}{2}\right)z^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)z^3 - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)z^4 + \dots;$$

$$1 - \frac{3^2+3}{4(2!)}z^2 + \frac{3^4+3}{4(4!)}z^4 - \frac{3^6+3}{4(6!)}z^6 + \dots;$$

$$1 - z + z^4 - z^5 + z^8 - z^9 + z^{12} - z^{13} + \dots;$$

$$(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \dots;$$

$$1 - \frac{z^2}{2^2(m+1)} + \frac{z^4}{2^4(2!)(m+1)(m+2)} - \frac{z^6}{2^6(3!)(m+1)(m+2)(m+3)} + \dots;$$

$$\frac{z^2}{2^2} - \frac{z^4}{2^4(2!)^2}\left(1 + \frac{1}{2}\right) + \frac{z^6}{2^6(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \dots;$$

$$1 + mz + \frac{m(m-1)}{2!}z^2 + \frac{m(m-1)(m-2)}{3!}z^3 + \dots;$$

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}z^3 + \dots.$$

247–253. The last series above is known as the hypergeometric series. It should be carefully studied. The function

to which it converges can be represented by $F(\alpha, \beta, \gamma, z)$. Assuming Maclauren's formula from calculus, prove:

$$F(1, 1, 2, -x) = \frac{1}{x}\log(1+x);$$

$$F\left(\frac{1}{2}, 1, \frac{3}{2}, x^2\right) = \frac{1}{2x}\log\frac{1+x}{1-x};$$

$$xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) = \arcsin x;$$

$$xF\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right) = \arctan x;$$

$$\lim_{\alpha\to\infty} F\left(\alpha, 1, 1, \frac{x}{\alpha}\right) = e^x;$$

$$\lim_{\alpha\to\infty} xF\left(\alpha, \alpha, \frac{3}{2}, -\frac{x^2}{4\alpha^2}\right) = \sin x;$$

$$\lim_{\alpha\to\infty} F\left(\alpha, \alpha, \frac{3}{2}, \frac{x^2}{4\alpha^2}\right) = \sinh x.$$

254. Show how to get power series for $\cos x$ and for $\cosh x$ from the hypergeometric function $F(\alpha, \beta, \gamma, x)$.

255. Show that $F(\alpha, \beta, \gamma, x)$ satisfies the differential equation

$$x(x-1)\frac{d^2y}{dx^2} + \left((\alpha+\beta+1)x - \gamma\right)\frac{dy}{dx} + \alpha\beta y = 0.$$

256. Functions known as Bessel's functions can be defined by the following series:

$$J_n(x) = x^n \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{2^{n+2s}(s!)\left((n+s)!\right)}.$$

Show that they satisfy the following relation:

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x).$$

257–259. Assuming Maclauren's formula, obtain power series expansions for the following functions. Examine each series for convergence. $\log(1+x)$, e^x, $\sin x$.

260–275. By means of the theorems of this and previous chapters and the series obtained in Exercises 257–259, obtain power series expansions for $\cos x$, $\tan x$, $\cosh x$, $\log(1-x)$, e^{2x}, $e^x \sin x$, $e^x \cosh x$, $\sin e^x$, $\cos \sin x$, $\log$ [illegible] x, $\arcsin x$, $\cosh^{-1} x$, $\frac{1}{1+x}$, $\frac{1}{1+x^2}$, $\arctan x$, $\frac{1}{e^x+1}$. Discuss convergence in each case. Obtain at least one of these series by inversion of a series already obtained.

276. Prove the following theorem. HYPOTHESES:

$$\text{(i)} \quad \sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

converge absolutely;

$$\text{(ii)} \quad f_1(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } f_2(x) = \sum_{n=0}^{\infty} b_n x^n.$$

CONCLUSION:

$$\sum_{n=0}^{\infty} a_n b_n = \frac{1}{2\pi}\int_0^{2\pi} f_1(e^{\theta i}) f_2(e^{\theta i})\, d\theta.$$

Definition. A series of the type $\sum_{n=1}^{\infty} a_n \frac{z^n}{1-z^n}$ is known as a Lambert series. We say that it corresponds to the power series $\sum_{n=1}^{\infty} a_n z^n$.

277. Prove the following theorem. HYPOTHESES:

$$\text{(i)} \quad \sum_{n=1}^{\infty} a_n \text{ diverges; (ii)} \quad \sum_{n=1}^{\infty} a_n z^n$$

has a circle of convergence of radius R. CONCLUSION:

$$\sum_{n=1}^{\infty} a_n \frac{z^n}{1-z^n}$$

converges at all points, $|z| \neq 1$, where $\sum_{n=1}^{\infty} a_n z^n$ converges and conversely. It converges uniformly over any circle of fixed radius $r < R$.

[HINT: Write the series in the form

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n \frac{z^n}{1-z^n}(1-z^n)$$

and

$$\sum_{n=1}^{\infty} a_n \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} a_n z^n \cdot \frac{1}{1-z^n}$$

and apply Theorems 119, 120.]

278. Prove the theorem. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n$ converges.

CONCLUSION: $\sum_{n=1}^{\infty} a_n \frac{z^n}{1-z^n}$ converges at all points where $|z| \neq 1$ and uniformly over any fixed circle of radius $\rho < 1$.

CHAPTER XII

DIRICHLET SERIES

§ 1. Convergence theorems.

Definition 31. *A series of the type*

$$(1) \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

where $\lambda_n < \lambda_{n+1}$ *are real,* and where* $\lambda_n \to \infty$, a_n *being independent of* z, *is known as a Dirichlet series in* z.

Particular types of Dirichlet series are obtained by assigning λ_n. For example, we can let $\lambda_n = \log n$.

Definition 32. *A series of the type,*

$$(2) \qquad \sum_{n=1}^{\infty} a_n n^{-z}$$

is known as an ordinary Dirichlet series.

If we let $\lambda_n = n$ and make the transformation $w = e^{-z}$, (1) goes over into the power series $\sum_{n=1}^{\infty} a_n w^n$.

Due to their generality and importance a study of Dirichlet series will be made somewhat like that made of power series. We begin with a fundamental convergence theorem.

Theorem 163. HYPOTHESIS: *Series* (1) *converges at*

$$z_0 = x_0 + y_0 i.$$

CONCLUSION: (1) *converges uniformly over the angular region defined by*

$$|\,am\,(z - z_0)\,| \leqq \omega,$$

$$0 \leqq \omega < \frac{\pi}{2}.$$

* This assumption is usually made, although it is possible to develop an extensive theory without it.

PROOF: Replace z by $z' + z_0$. Series (1) becomes

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z_0} e^{-\lambda_n z'} = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z'}$$

which is of the form (1), converging when $z' = 0$. We go back then, and assume, without loss of generality, that $z_0 = 0$.

$$e^{-\lambda_n z} - e^{-\lambda_{n+1} z} = z \int_{\lambda_n}^{\lambda_{n+1}} e^{-uz}\,du.$$

And, when x, the real part of z, is positive, which we assume,

$$(3) \qquad |e^{-\lambda_n z} - e^{-\lambda_{n+1} z}| \leqq |z| \int_{\lambda_n}^{\lambda_{n+1}} e^{-ux}\,du$$

$$= \frac{|z|}{x}(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}).$$

Next, as usual, let

$$s_n(z) = \sum_{n=1}^{n} a_n e^{-\lambda_n z} \text{ and } f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}.$$

Let $\frac{|y|}{x} \leqq \tan\omega = c$, thus confining z to the angular region in question. Let an $\epsilon > 0$ be given. Then, as is readily verified,

$$\sum_{n=\nu}^{\infty} a_n e^{-\lambda_n (x+yi)}$$

$$= \sum_{n=\nu}^{\infty} [s_n(0) - f(0)]\,[e^{-\lambda_n (x+yi)} - e^{-\lambda_{n+1}(x+yi)}]$$

$$-\big(s_{\nu-1}(0) - f(0)\big)\, e^{-\lambda_\nu (x+yi)}.$$

Let ν be so large that $\lambda_\nu > 0$ and $|s_n(0) - f(0)| < \eta$, whenever $n \geqq \nu - 1$. Then

$$\left|\sum_{n=\nu}^{\infty} a_n e^{-\lambda_n (x+yi)}\right| < \eta \sum_{n=\nu}^{\infty} |e^{-\lambda_n (x+yi)} - e^{-\lambda_{n+1}(x+yi)}| + \eta e^{-\lambda_\nu x}.$$

By (3) this is less than

$$\eta \cdot \frac{|x+yi|}{x} \sum_{n=\nu}^{\infty} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) + \eta$$

$$= \eta \sqrt{1 + \left(\frac{y}{x}\right)^2} \cdot e^{-\lambda_\nu x} + \eta < \eta\sqrt{1+c^2} + \eta \leqq 2\eta\sqrt{1+c^2} < \epsilon,$$

if $\eta < \frac{\epsilon}{2\sqrt{1+c^2}}$; establishing the theorem.

Corollary 1. (1) *converges uniformly over the complete angular region described in the theorem, including its vertex and bounding lines.*

Corollary 2. $f(z)$ *approaches* $f(z_0)$ *when* z *approaches* z_0 *in any manner, so long as it lies in the angular region of Corollary* 1.

Theorem 164. *Series* (1) *either converges for all values of* z *or diverges for all values of* z, *or there exists a line in the complex plane,* $x = a$, *such that* (1) *converges at all points to the right of this line and diverges at all points to the left.*

PROOF: An example of a Dirichlet series that converges for all values of z is $\sum_{n=1}^{\infty} \frac{1}{n!} n^{-z}$, of one that diverges for all values of z is $\sum_{n=1}^{\infty} n!\, n^{-z}$, of one that converges for certain values of z and diverges for other values is $\sum_{n=1}^{\infty} n^{-z}$.

In the third case the existence of the line $x = a$ is readily established as in the corresponding theorem for power series (Theorem 135). Details are omitted.

Definition 32. *The value* a *of Theorem* 164 *is called the abscissa of convergence of series* (1).

Theorem 165. HYPOTHESIS: *Series* (1) *converges at* $z = 0$.
CONCLUSION: (1) *converges uniformly throughout the angular region bounded by the two curves*

$$y = e^{Mx} - 1 \quad \text{and} \quad y = -(e^{Mx} - 1),$$

*M being any positive number.**

As in Theorem 164, there is no loss of generality in our assuming the known point of convergence to be the origin.

* For a general discussion of the region of uniform convergence see L. Neder, *Mathematische Zeitschrift* 15, S. 286.

PROOF: Let c_1 be a constant, such that $|s_n(0)| < c_1$ for all values of n. Then, for $N \geqq 1$ and $x > 0$, we have

$$f(z) - s_N(z) = \sum_{n=N+1}^{\infty} (s_n(0) - s_{n-1}(0))\, e^{-\lambda_n z}$$

$$= \sum_{n=N+1}^{\infty} s_n(0)\,(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) - s_N(0)\, e^{-\lambda_{N+1} z},$$

$$|f(z) - s_N(z)| \leqq c_1 \sum_{n=N+1}^{\infty} |e^{-\lambda_n z} - e^{-\lambda_{n+1} z}| + c_1 e^{-\lambda_{N+1} x}$$

$$\leqq c_1 \frac{|z|}{x} \sum_{n=N+1}^{\infty} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) + c_1 e^{-\lambda_{N+1} x}$$

$$= c_1 \frac{|z|}{x} e^{-\lambda_{N+1} x} + c_1 e^{-\lambda_{N+1} x} \leqq 2 c_1 \frac{|z|}{x} e^{-\lambda_{N+1} x}.$$

Moreover, with z in the region in question,

$$|z| = |x + yi| \leqq x + |y| \leqq x + e^{Mx} - 1$$

$$< \frac{Mx}{M} + e^{Mx} < \frac{e^{Mx}}{M} \cdot e^{Mx} < c_2 e^{Mx}.$$

Hence

$$|f(z) - s_N(z)| < 2 c_1 c_2 \frac{e^{Mx}}{x} e^{-\lambda_{N+1} x}$$

$$= c_3 \frac{e^{(M - \lambda_{N+1}) x}}{x} = c_3 \cdot \frac{1}{x} \cdot \frac{1}{e^{(\lambda_{N+1} - M) x}}.$$

If an $\epsilon > 0$ is given, determine N_0 so that $\lambda_{N_0+1} > M$ and take X so great that

$$c_3 \cdot \frac{1}{X} \cdot \frac{1}{e^{(\lambda_{N_0+1} - M) x}} < \epsilon.$$

We then have, for values of $x \geqq X$ and $N > N_0$,

$$|f(z) - s_N(z)| < \epsilon.$$

The convergence is, then, uniform over the portion of the angular region in question where $x \geqq X$. The portion where $x \leqq X$ is covered by Theorem 163. Consequently, by Theorem 99, convergence is uniform over the whole region.

For continuity, integrability, differentiability, &c., of Dirichlet series we now refer to theorems of chap. IX.

Theorem 166. *Series* (1) *either converges absolutely for no value of $z = x + yi$, or converges absolutely for all values of z, or there exists a real number b, such that when $x \geqq b$* (1) *converges absolutely and uniformly, and when $x < b$* (1) *does not converge absolutely.*

PROOF: Assume that the series converges absolutely when $z = z_0$. When $\lambda_n \geqq 0$ and $x \geqq x_0$, which we assume,

$$|a_n e^{-\lambda_n z}| = |a_n| e^{-\lambda_n x} = |a_n| e^{-\lambda_n x_0} e^{-\lambda_n (x - x_0)} \leqq |a_n| e^{-\lambda_n x_0}.$$

Absolute uniform convergence follows over the half-plane where $x \geqq x_0$. If (1) does not converge absolutely when $z = z_0$, we readily show in like manner that it does not converge absolutely at any point where $x < x_0$.

If (1) converges absolutely at one point and does not converge absolutely at another the existence of b is readily established. The existence of such series is proved by the single example

$$\sum_{n=1}^{\infty} (-1)^n n^{-z}.$$

The series given under Theorem 164 also serve to illustrate the other cases under this theorem.

Definition 33. *b is called the abscissa of absolute convergence of series* (1).

The whole topic is illuminated by the following additional examples. a and b are used to denote the abscissas of convergence and absolute convergence respectively. We use the symbolism $a = \infty$ to mean that the series converges at no point. Similarly we use the symbolism as $a = -\infty$, $b = \infty$, and $b = -\infty$ with obvious import.

EXAMPLES

1. $a = b = -\infty$

$$\sum_{n=1}^{\infty} \frac{1}{n!} n^{-z}.$$

2. $-\infty < a < b < \infty$

$$\sum_{n=2} \frac{(-1)^n}{n} \cdot \frac{1}{(\log n)^z}.$$

3. $-\infty = a < b = \infty$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \frac{1}{(\log n)^z}.$$

4. $-\infty < a = b < \infty$

$$\sum_{n=1}^{\infty} \frac{1}{n^z}.$$

5. $-\infty < a < b < \infty$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}.$$

6. $-\infty < a < b = \infty$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(\log n)^z}.$$

7. $a = b = \infty$

$$\sum_{n=1}^{\infty} \frac{n!}{n^z}.$$

§ 2. Superior limits.

This section is a digression from the topic of Dirichlet series. It is preparatory to the succeeding section and can be considered as in the nature of a foot-note or lemma.

Definition 34. *S is a limiting number* of a set of numbers, X, if for any $\epsilon > 0$ there are an infinite number of different numbers of X fulfilling the conditions, $x \neq S$ and $|x - S| < \epsilon$ simultaneously.*

The limiting numbers of the set X form a set which we designate by L. If X has no limiting number we still speak of its set of limiting numbers, the set being vacuous or empty. This is a convenient mode of expression.

* In geometric language: limiting point.

Theorem 167. HYPOTHESES: (i) *X is infinite*; (ii) *all numbers of X are real and lie on the interval*

$$(1) \qquad G_1 \leqq x \leqq G_2.$$

CONCLUSIONS: (i) *The set L contains at least one number*; (ii) *the numbers of L lie on the interval* (1); (iii) *L contains a maximum number.*

PROOF: Divide (1) into two equal closed* portions. At least one of these portions must contain an infinite number of the numbers of X. Divide this into closed halves and proceed. The process is a familiar one. The intervals shut down on a number which is a limiting number of X.

That all numbers of L lie on (1) is only noted. See the proofs of Theorems 7 and 8.

Moreover, L contains all its limiting numbers. For, let P be a limiting number of L. Then denote a monotonic sequence of different numbers from L approaching P by L_n. For definiteness suppose this an increasing sequence. Now from the set X choose x_1 such that $|x_1-L_1| < \frac{1}{2}$. Then choose $x_2 > x_1$ such that $|x_2-L_2| < \frac{1}{4}$. Choose $x_3 > x_2$ such that $|x_3-L_3| < \frac{1}{8}$, &c. This sequence has P for limit, showing that P is also a number of L.

We next prove that there is a number of L which is a maximum. As we have done once already in this proof, divide (1) into closed halves. Consider that half which contains numbers of L and which, on the linear scale, is chosen as far to the right as possible. If this half contains only a finite number of numbers of L, the largest is the desired maximum. If not, divide the half considered into two equal portions and proceed as before. The intervals shut down on a number which is a limiting number of L and hence itself a member of that set. It is a member than which no member is greater, and hence is the desired maximum.

Definition 35. *The maximum of the set L is called the superior limit of X.*

If the points of X do not remain finite to the right we can

* The point of division is considered to belong to each portion. It is thus counted twice.

introduce the symbolic limit ∞, and say that every infinite set of real numbers has a superior limit.

In case the numbers of X form a sequence, s_n, we frequently write, $\underset{n\to\infty}{\text{superior limit}}\, s_n$.

A definition and discussion of inferior limits entails only formal modifications of the preceding.

The terms upper limit and lower limit are frequently used for what we have called superior limit and inferior limit respectively. These terms are not to be confused, however, with upper boundary and lower boundary, which are usually used with a different significance.

A number, B, is called the upper boundary of X if all numbers of X are less than or equal to B and if for any $\epsilon > 0$ there exists at least one number of X satisfying the condition $B-x < \epsilon$. This number may be B itself.

§ 3. Values of the abscissas of convergence.

The next problem is to determine the values of the convergence abscissas, a and b.

We have shown how the essential character of the series is not changed by the transformation $z = z'+z_0$. Consequently, in case $a \neq -\infty$ there is no loss of generality in our assuming that it is positive.* This we proceed to do.

Theorem 168. HYPOTHESES: (i) *a is the abscissa of convergence of*

$$(1) \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n z};$$

(ii) $0 < a < \infty$. CONCLUSION:

$$\underset{n\to\infty}{\text{superior limit}} \frac{\log\left|\sum_{n=1}^{n} a_n\right|}{\lambda_n} = L$$

exists and equals a.

* In case $a < 0$ in the formula for L replace $\sum_{n=1}^{n} a_n$ by $\sum_{n=n+1}^{\infty} a_n$. For references to the literature see: 'Théorie Générale des Series de Dirichlet.' G. Valiron: *Mémorial des Sciences Mathématiques.*

PROOF: We shall make the proof by proving, I, that if (1) converges when $z = x_0 > 0$, then

$$\underset{n\to\infty}{\text{superior limit}} \frac{\log \left| \sum_{n=1}^{n} a_n \right|}{\lambda_n}$$

exists and is no larger than x_0; and II, that when $x > L$ (1) converges.

I. Assume (1) convergent when $z = x_0 > 0$. Let $s_n(z)$ have the usual significance.

$$\begin{aligned} s_n(0) &= \sum_{n=1}^{n} a_n e^{-\lambda_n x_0} e^{\lambda_n x_0} \\ &= \sum_{n=2}^{n} \left(s_n(x_0) - s_{n-1}(x_0)\right) e^{\lambda_n x_0} + s_1(x_0)\, e^{\lambda_1 x_0} \\ &= \sum_{n=1}^{n-1} s_n(x_0)\left(e^{\lambda_n x_0} - e^{\lambda_{n+1} x_0}\right) + s_n(x_0)\, e^{\lambda_n x_0}. \end{aligned}$$

Suppose $|s_n(x_0)| < B$ for all values of n. Then

$$\begin{aligned} |s_n(0)| &< B \sum_{n=1}^{n-1} \left(e^{\lambda_{n+1} x_0} - e^{\lambda_n x_0}\right) + B e^{\lambda_n x_0} \\ &= B\left(e^{\lambda_n x_0} - e^{\lambda_1 x_0}\right) + B e^{\lambda_n x_0} < 2 B e^{\lambda_n x_0}. \end{aligned}$$

Moreover, for $\delta > 0$ and all sufficiently great values of n, $e^{\lambda_n \delta} > 2B$. Consequently $|s_n(0)| < e^{\lambda_n (x_0 + \delta)}$ or

$$x_0 + \delta > \frac{\log |s_n(0)|}{\lambda_n}. \tag{2}$$

In case $s_n(0) = 0$ for a particular value of n, of course $\log |s_n(0)|$ does not exist. But, as the series (1) diverges when $z = 0$, there are an infinite number of values of n for which $s_n(0) \neq 0$. In case $s_n(0) = 0$, we write symbolically $\log |s_n(0)| = -\infty$ and understand (2) to hold symbolically in this case also. We immediately infer that

$$\underset{n\to\infty}{\text{superior limit}} \frac{\log |s_n(0)|}{\lambda_n}$$

exists and that it is less than or equal to x_0.

II. We assume $x > L$.

Let $\delta > 0$ be given. Choose M so large that when $n > M$, $\lambda_n > 0$.

If $M \geqq 1$ is large enough,

$$\frac{\log |s_n(0)|}{\lambda_n} < L + \frac{\delta}{2},$$

when $n > M$ and $s_n(0) \neq 0$. Whether $s_n(0) = 0$ or not,

$$|s_n(0)| < e^{\lambda_n \left(L + \frac{\delta}{2}\right)}.$$

Moreover, when $n \geqq 2$,

$$\begin{aligned} \sum_{n=v}^{w} a_n e^{-\lambda_n z} &= \sum_{n=v}^{w} \left(s_n(0) - s_{n-1}(0)\right) e^{-\lambda_n z} \\ &= \sum_{n=v}^{w} s_n(0)\left(e^{-\lambda_n z} - e^{\lambda_{n+1} z}\right) - s_{v-1}(0)\, e^{-\lambda_v z} + s_w(0)\, e^{-\lambda_{w+1} z}. \end{aligned}$$

Let $w \geqq v \geqq M + 1$. Then

$$\begin{aligned} \left| \sum_{n=v}^{w} a_n e^{-\lambda_n (L+\delta)} \right| &< \sum_{n=v}^{w} e^{\lambda_n \left(L + \frac{\delta}{2}\right)} \left(e^{-\lambda_n (L+\delta)} - e^{-\lambda_{n+1}(L+\delta)}\right) \\ &\quad + e^{\lambda_{v-1}\left(L + \frac{\delta}{2}\right) - \lambda_v (L+\delta)} + e^{\lambda_w \left(L + \frac{\delta}{2}\right) - \lambda_{w+1}(L+\delta)} \\ &< \frac{1}{L+\delta} \sum_{n=v}^{w} e^{\lambda_n \left(L + \frac{\delta}{2}\right)} \int_{\lambda_n}^{\lambda_{n+1}} e^{-u(L+\delta)}\, du \\ &\quad + e^{\lambda_{v-1}\left(L + \frac{\delta}{2}\right) - \lambda_v (L+\delta)} + e^{\lambda_w \left(L + \frac{\delta}{2}\right) - \lambda_{w+1}(L+\delta)} \\ &< \frac{1}{L+\delta} \sum_{n=v}^{w} \int_{\lambda_n}^{\lambda_{n+1}} e^{u\left(L + \frac{\delta}{2}\right)} e^{-u(L+\delta)}\, du + e^{-\lambda_v \frac{\delta}{2}} + e^{-\lambda_{w+1}\frac{\delta}{2}} \\ &= \frac{1}{L+\delta} \int_{\lambda_v}^{\lambda_{w+1}} e^{-u\frac{\delta}{2}}\, du + e^{-\lambda_v \frac{\delta}{2}} + e^{-\lambda_{w+1}\frac{\delta}{2}} \\ &= \frac{2}{\delta(L+\delta)} \left[e^{-\lambda_v \frac{\delta}{2}} - e^{-\lambda_{w+1}\frac{\delta}{2}}\right] + e^{-\lambda_v \frac{\delta}{2}} + e^{-\lambda_{w+1}\frac{\delta}{2}}, \end{aligned}$$

which $\to 0$ when $v \to \infty$ and hence also $w \to \infty$, establishing the theorem. [See Theorem 21.]

Corollary. *In case the abscissa of absolute convergence is positive, its value, b, is given by*

$$b = \underset{n\to\infty}{\text{superior limit}} \frac{\log \sum_{n=1}^{n} |a_n|}{\lambda_n}.$$

Theorem 169. HYPOTHESES: (i) *As previously, a and b are respectively the abscissas of convergence and of absolute convergence of* (1); (ii) superior limit$_{n\to\infty}$ $\dfrac{\log n}{\lambda_n} = l$ *exists.* CONCLUSION: $b - a \leqq l$.

PROOF: We shall proceed to prove that if (1) converges absolutely when $z = x_0$ it converges absolutely when

$$z = x_0 + l + \delta, \quad \delta > 0.$$

We prove this by reference to Theorem 38 and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{l+\delta}{l+\delta/2}}}$$

which converges. In other words, we shall prove that

$$a_n e^{-\lambda_n (x_0 + l + \delta)} n^{\frac{l+\delta}{l+\delta/2}}$$

remains finite, whereupon the theorem will have been proved.

$$a_n e^{-\lambda_n (x_0 + l + \delta)} n^{\frac{l+\delta}{l+\delta/2}} = a_n e^{-\lambda_n (x_0 + l + \delta)} e^{\frac{l+\delta}{l+\delta/2} \log n}.$$

Since $\log n < \lambda_n \left(l + \dfrac{\delta}{2}\right)$ for sufficiently great values of n by the hypothesis,

$$a_n e^{-\lambda_n (x_0 + l + \delta)} n^{\frac{l+\delta}{l+\delta/2}} < a_n e^{-\lambda_n (x_0 + l + \delta)} \cdot e^{(l+\delta)\lambda_n}$$
$$= a_n e^{-\lambda_n x_0} \longrightarrow 0,$$

due to the convergence of the series at x_0.

Corollary 1. *When $\lambda_n = n$, $l = 0$, a well-known result if the Dirichlet series is transformed into the power series by means of the substitution $w = e^{-z}$.*

Corollary 2. *In case $\lambda_n = \log n$, $l = 1$.*

§ 4. Uniqueness theorems.

Theorem 170. HYPOTHESIS:

$$\text{(1)} \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = 0$$

for all values of $x \geqq \gamma$. CONCLUSION: $a_1 = a_2 = \ldots = 0$.

PROOF: Assume $\gamma = 0$ in which again there is no loss of generality. (1) and hence

$$\text{(2)} \qquad \sum_{n=1}^{\infty} a_n e^{-(\lambda_n - \lambda_1)x}$$

converges uniformly to zero over the positive half of the axis of reals.

Suppose $a_1 \neq 0$. If a fixed N is chosen large enough, for all values of $x \geqq 0$

$$\left| \sum_{n=1}^{N} a_n e^{-(\lambda_n - \lambda_1)x} \right| < \left| \frac{a_1}{2} \right|,$$

that is

$$\left| a_1 + \sum_{n=2}^{N} a_n e^{-(\lambda_n - \lambda_1)x} \right| < \left| \frac{a_1}{2} \right|,$$

or

$$\left| \sum_{n=2}^{N} a_n e^{-(\lambda_n - \lambda_1)x} \right| > \left| \frac{a_1}{2} \right|.$$

But by taking x large enough each term of (2), and hence the sum of any finite number of terms, can be made as close to zero as desired. That is if x is large enough,

$$\left| \sum_{n=2}^{N} a_n e^{-(\lambda_n - \lambda_1)x} \right| < \left| \frac{a_1}{2} \right|,$$

a contradiction. Hence $a_1 = 0$. A repetition shows that $a_2 = 0$, and mathematical induction that $a_n = 0$.

Corollary. HYPOTHESIS:

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \sum_{n=1}^{\infty} b_n e^{-\lambda_n x}$$

when $x \geqq \gamma$. CONCLUSION: $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$, $\ldots$.

An examination of the proof of this theorem shows that it will result in the following more general theorem.

Theorem 171. HYPOTHESIS:

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = 0$$

for particular values of x greater than any M whatsoever.
CONCLUSION: $a_1 = a_2 = \ldots = a_n = \ldots = 0$.

The next theorem is an extension to the complex domain.

Theorem 172. HYPOTHESES: (i)

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges when $z = z_0 = x_0 + y_0 i$; (ii) *M is any positive number*; (iii) $f(z) = 0$ *at an infinite number of points, in the region S defined by* $x \geqq x_0$, $z_n = x_n + y_n i$, $n = 1, 2, \ldots$, *and* $-e^{M(x-x_0)} + 1 \leqq y \leqq e^{M(x-x_0)} - 1$; (iv) $x_n \to \infty$. CONCLUSION: $a_1 = a_2 = \ldots = 0$.

PROOF: Let us assume $a_1 = a_2 = \ldots = a_{k-1} = 0$ but $a_k \neq 0$. We proceed to show that for values of $x > X$ (some constant) $f(z) \neq 0$. This will be a contradiction and the proof will be complete.

We shall consider

$$(3) \qquad \sum_{n=k}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} = a_k + \sum_{n=k+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z}.$$

Due to the uniform convergence of the series there exists an N, such that

$$\left| \sum_{n=N+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| < \frac{|a_k|}{2}$$

for all z's of S simultaneously. Hence

$$\left| \sum_{n=k+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| \leqq \left| \sum_{n=k+1}^{N} a_n e^{-(\lambda_n - \lambda_k) z} \right| + \frac{|a_k|}{2}.$$

Hold N fast.

$$\left| \sum_{n=k+1}^{N} a_n e^{-(\lambda_n - \lambda_k) z} \right| \leqq \sum_{n=k+1}^{N} |a_n| e^{-(\lambda_n - \lambda_k) x}.$$

Now we can take X so large that, when $x > X$, each term

in the right-hand member is less than $\frac{1}{N-k} \cdot \frac{|a_k|}{2}$ and hence the right-hand member itself is less than $\frac{|a_k|}{2}$. Then

$$\left| \sum_{n=k+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| < |a_k|$$

or

$$|a_k| - \left| \sum_{n=k+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| > 0.$$

But, from (3),

$$\left| \sum_{n=k}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| \geqq |a_k| - \left| \sum_{n=k+1}^{\infty} a_n e^{-(\lambda_n - \lambda_k) z} \right| > 0.$$

Factoring out $|e^{-\lambda_k z}| \neq 0$, we have

$$\left| \sum_{n=k}^{\infty} a_n e^{-\lambda_n z} \right| > 0.$$

This is when $x > X$.

Corollary. *For every Dirichlet series converging to a sum, $f(z)$, not identically zero, an X can be found corresponding to any $M > 0$, such that, when $x > X$ and $|y| \leqq e^{Mx} - 1$,*

$$f(z) \neq 0.$$

The only reason that y was limited in Theorem 172 was that z be restricted to a region of uniform convergence. In the case of an absolutely convergent series the restriction on y is no longer necessary and we have the following theorem.

Theorem 173. HYPOTHESES: (i) $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ *converges absolutely when* $x > b$; (ii) $f(z) \not\equiv 0$. CONCLUSION: *There exists a number X such that $f(z) \neq 0$ when $x \geqq X$.*

§ 5. Multiplication of series by the Dirichlet rule.

In chapter VII, section 2, we considered in some detail the multiplication of series. The method employed there can be described as the power series rule. If $\sum_{n=0}^{\infty} a_n x^n$ and

$\sum_{n=0}^{\infty} b_n x^n$ are multiplied together formally, all terms of like power in x collected, and the resulting series denoted by $\sum_{n=0}^{\infty} c_n x^n$, reference will show that c_n is the same as the c_n of that section

It is now proposed to form a generalization of this rule based on Dirichlet series. We shall proceed at first in a purely formal manner. Let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \quad \text{and} \tag{1}$$

$$g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}. \tag{2}$$

We assume that the λ's are the same in each series. If this is not the case, take all λ's of both series and arrange in order of increasing magnitude. In case the same number occurs in both series it is taken but once. Designate these numbers in the order just determined as $\lambda_1, \lambda_2, \ldots$. In case λ_k does not occur in series (1) we insert the term with zero coefficient, &c. Any two series as (1) and (2) can then be written with the same sequence of λ's and our assumption is justified. We proceed formally to multiply together these two series, collecting all coefficients of e to each occurring power. We have a series $\sum_{\gamma=1}^{\infty} c_\gamma e^{-\nu_\gamma z}$, where $c_\gamma = \sum a_l b_m$ where $\lambda_l + \lambda_m = \nu_\gamma$. As in the case of power series we can consider only the series of coefficients, if we like, and speak of the Dirichlet product of

$$A = \sum_{n=1}^{\infty} a_n \quad \text{and} \tag{3}$$

$$B = \sum_{n=1}^{\infty} b_n \tag{4}$$

according to the particular succession $\lambda_1, \lambda_2, \lambda_3, \ldots$, obtaining the result $\sum_{n=1}^{\infty} c_n$.

Theorem 174. HYPOTHESIS: (3) *converges absolutely and* (4) *converges.* CONCLUSION: $\Sigma a_m b_n$ *converges to the value* AB *if the terms* $a_m b_n$ *are arranged so that* $\lambda_m + \lambda_n$ *does not decrease.*

PROOF: Let $s_n = \sum_{n=1}^{n} a_n$, $\sigma_n = \sum_{n=1}^{n} b_n$, $\tau_n = \sum_{n=1}^{n} |a_n|$, and $S_\nu = \Sigma a_m b_n$ with $m \leqq \nu$ and $n \leqq \nu$.

$$S_n = \sum_{l=1}^{n} a_l (b_1 + b_2 + \ldots + b_{\psi(n,l)}).$$

We simply group all terms containing a_l together.

Now let any $\epsilon > 0$ be given. We shall show that there exists an M, such that when we have included the M^2 terms $a_l b_m$, $l = 1, \ldots, M$; $n = 1, \ldots, M$ in S_n the relation

$$|S_n - AB| < \epsilon$$

holds; whereupon the proof will be complete.

There exists a G such that both $\tau_n < G$ and $|s_n| < G$ for all values of n. Now choose M so great that for all values of $w > v > M$,

$$\sum_{n=v}^{w} |a_n| < \frac{\epsilon}{6G} \quad \text{and} \tag{5}$$

$$\left| \sum_{n=v+1}^{w} b_n \right| < \frac{\epsilon}{3G}, \tag{6}$$

and in addition so that

$$|s_n \sigma_n - AB| < \frac{\epsilon}{3}, \quad n > M.$$

(See Theorems 13 and 21.) Next take n so much greater than $\xi > M$ that

$$\psi(n, l) \geqq \xi, \quad l = 1, 2, \ldots, \xi.$$

$$S_n = \sum_{l=1}^{n} a_l \sigma_{\psi(n,l)},$$

$$(7)\quad S_n - s_n\sigma_n = \sum_{l=1}^{n} a_l(\sigma_{\psi(n,l)} - \sigma_n)$$
$$= \sum_{l=1}^{\xi} a_l(\sigma_{\psi(n,l)} - \sigma_n) + \sum_{l=\xi+1}^{n} a_l(\sigma_{\psi(n,l)} - \sigma_n).$$

From (5), (6), and (7),

$$|S_n - s_n\sigma_n| \leqq \frac{\epsilon}{3G}\sum_{l=1}^{\xi}|a_l| + 2G\sum_{l=\xi+1}^{n}|a_l|$$
$$< \frac{\epsilon}{3G}G + 2G\frac{\epsilon}{6G} = \frac{2}{3}\epsilon.$$

Hence

$$|S_n - AB| \leqq |S_n - s_n\sigma_n| + |s_n\sigma_n - AB| < \frac{2}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.$$

The following two theorems are corollaries, but due to their importance are given independent statement.

Theorem 175. HYPOTHESES: (i) (3) *converges absolutely*; (ii) (4) *converges*; (iii) $c_n = \sum_{\lambda_l+\lambda_m=\nu_n} a_l b_m$. CONCLUSION:

$$\sum_{n=1}^{\infty} c_n = AB.$$

Theorem 176. HYPOTHESES:

$$\text{(i)}\quad f_1(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges absolutely when $z = z_0$;

$$\text{(ii)}\quad f_2(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$$

converges when $z = z_0$. CONCLUSION: *The product series* $\sum_{n=1}^{\infty} c_n e^{-\nu_n z}$ *converges at* $z = z_0$ *to* $f_1(z_0)\cdot f_2(z_0)$.

The next theorem is proved at this point primarily as a lemma for the theorem which is to follow. The notation employed will be a little different from anything previously used, but it is not believed that an explanation is necessary.

Theorem 177. HYPOTHESES: (i) (3) *and* (4) *converge to* A *and* B *respectively*; (ii) $c_n = \sum_{\lambda_l+\lambda_m=\nu_n} a_l b_m$; (iii) $\sum_{\nu_n \leqq x} c_n = c(x)$.

CONCLUSION: $\lim_{x\to\infty} \frac{1}{x}\int_{\blacksquare}^{x} c(x)\,dx$ *exists and equals* AB.

PROOF: Let $s(x) = \sum_{\lambda_n \leqq x} a_n$ and $\sigma(x) = \sum_{\lambda_n \leqq x} b_n$. Then

$$c(x) = \sum_{\lambda_l+\lambda_m \leqq x} a_l b_m = \sum_{\lambda_l \leqq x-\lambda_1} a_l \sum_{\lambda_m \leqq x-\lambda_l} b_m = \sum_{\lambda_l \leqq x-\lambda_1} a_l \sigma(x-\lambda_l).$$

If $y < \lambda_1$, let $\sigma(y) = 0$, then $\sigma(x-\lambda_l) = 0$ when $\lambda_l > x - \lambda_1$. The last sum, consequently, can be extended to larger values of λ_l, if desired, without changing its value. Integrating and letting $x > 2\lambda_1$, we have

$$\int_{2\lambda_1}^{x} c(x)\,dx = \int_{2\lambda_1}^{x} \sum_{\lambda_l \leqq x-\lambda_1} a_l\sigma(x-\lambda_l)\,dx$$
$$= \sum_{\lambda_l \leqq x-\lambda_1} a_l \int_{\lambda_l+\lambda_1}^{x} \sigma(x-\lambda_l)\,dx = \sum_{\lambda_l \leqq x-\lambda_1} a_l \int_{\lambda_1}^{x-\lambda_l} \sigma(y)\,dy.$$

Moreover,

$$\int_{\lambda_1}^{x-\lambda_1} s(y)\sigma(x-y)\,dy = \int_{\lambda_1}^{x-\lambda_1}\Big[\sigma(x-y)\sum_{\lambda_l \leqq y} a_l\Big]dy$$
$$= \sum_{\lambda_l \leqq x-\lambda_1} a_l \int_{\lambda_l}^{x-\lambda_1} \sigma(x-y)\,dy = \sum_{\lambda_l \leqq x-\lambda_1} a_l \int_{\lambda_1}^{x-\lambda_l} \sigma(u)\,du.$$

Equating like values,

$$(8)\qquad \int_{2\lambda_1}^{x} c(y)\,dy = \int_{\lambda_1}^{x-\lambda_1} s(y)\sigma(x-y)\,dy.$$

We next write the identity

$$s(y)\sigma(x-y) = AB + (s(y)-A)B + A(\sigma(x-y)-B)$$
$$+ (s(y)-A)(\sigma(x-y)-B).$$

From this

$$\int_{\lambda_1}^{x-\lambda_1} s(y)\,\sigma(x-y)\,dy = AB\,(x-2\lambda_1) + B\int_{\lambda_1}^{x-\lambda_1} (s(y)-A)\,dy$$

$$+A\int_{\lambda_1}^{x-\lambda_1} (\sigma(x-y)-B)\,dy$$

$$+\int_{\lambda_1}^{x-\lambda_1} (s(y)-A)\,(\sigma(x-y)-B)\,dy.$$

But as $s(y) \to A$ and $\sigma(y) \to B$,

$$\frac{1}{x}\int_{\lambda_1}^{x-\lambda_1} (s(y)-A)\,dy \to 0 \quad \text{as} \quad x \to \infty;$$

$$\frac{1}{x}\int_{\lambda_1}^{x-\lambda_1} (\sigma(x-y)-B)\,dy \to 0 \quad \text{as} \quad x \to \infty;$$

$$\frac{1}{x}\int_{\lambda_1}^{x-\lambda_1} (s(y)-A)\,(\sigma(x-y)-B)\,dy \to 0 \quad \text{as} \quad x \to \infty.$$

Hence

$$\lim_{x\to\infty} \frac{1}{x}\int_{\lambda_1}^{x-\lambda_1} s(y)\,\sigma(x-y)\,dy = AB.$$

From this, by (8),

$$\lim_{x\to\infty} \frac{1}{x}\int_{2\lambda_1}^{x} c(y)\,dy = AB.$$

From this, since $\displaystyle\lim_{x\to\infty} \frac{1}{x}\int_{0}^{2\lambda_1} c(y)\,dy = 0,$

we have $$\lim_{x\to\infty} \frac{1}{x}\int_{0}^{x} c(y)\,dy = AB.$$

Theorem 178. HYPOTHESES: (3), (4), *and*

$$\sum_{n=1}^{\infty} c_n, \tag{9}$$

where $$c_n = \sum_{\lambda_l+\lambda_m=\nu_n} a_l b_m,$$

converge to A, B, and C respectively. CONCLUSION: $C = AB$.

PROOF: By the previous theorem

$$\lim_{x\to\infty} \frac{1}{x}\int_{0}^{x} (c(y)-C)\,dy = AB - C.$$

But $c(y) \to C$ as $y \to \infty$. From which one readily shows that

$$\frac{1}{x}\int_0^x (c(y)-C)\,dy \to 0 \quad \text{as} \quad x \to \infty.$$

That is $AB = C$.

As a corollary we have the following theorem.

Theorem 179. HYPOTHESIS:

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$$

and their product series $\displaystyle\sum_{n=1}^{\infty} c_n e^{-\mu_n z}$ *converge over a region S.* CONCLUSION: *Over S*

$$\sum_{n=1}^{\infty} c_n e^{-\mu_n z} = \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\right)\left(\sum_{n=1}^{\infty} b_n e^{-\lambda_n z}\right).$$

EXERCISES

279–294. Find the abscissas of convergence and of absolute convergence of each of the following series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n^{-z}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1)^{-z},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-nz}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} e^{-(2n-1)z},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(\log\log n)\,z}, \qquad \sum_{n=1}^{\infty} e^{-\log\log(2n-1)\,z},$$

$$\sum_{n=1}^{\infty} e^{-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)z}, \qquad \sum_{n=1}^{\infty} e^{-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n-1}\right)z},$$

$$1 - \frac{3^2+3}{4\,(2\,!)}\,2^{-z} + \frac{3^4+3}{4\,(4\,!)}\,4^{-z} - \frac{3^6+3}{4\,(6\,!)}\,6^{-z} + \dots,$$

$$1 - 1^{-z} + 4^{-z} - 5^{-z} + 8^{-z} - 9^{-z} + 12^{-z} - 13^{-z} + \dots,$$

$$\frac{e^{-2nz}}{2^2} - \frac{e^{-4nz}}{2^4\,(2\,!)^2}\left(1+\frac{1}{2}\right) + \frac{e^{-6nz}}{2^6\,(3\,!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \dots,$$

$$\sum_{n=1}^{\infty} 2^{-2^n z}, \qquad \sum_{n=1}^{\infty} a^n n^{-z}, \qquad \sum_{n=2}^{\infty} (\log n)^{-2} n^{-z},$$

$$\sum_{n=2}^{\infty} \left[(-1)^n + (\log n)^{-2}\right] n^{-z}, \qquad \sum_{n=2}^{\infty} \left[(-1)^n + (\log n)^{-2}\right] e^{-n^2 z}.$$

295. On page 159 are found examples of Dirichlet series with various abscissas of convergence and of absolute convergence. Give other examples illustrative of each case illustrated there. Do not choose any of these examples from the preceding set of series.

296. Multiply together the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^2}$$

by the Dirichlet series rule. Take successively $\lambda_n = n, \log n, n^2$. Draw conclusions as to the convergence of the product series.

297. Multiply other series together by the Dirichlet series rule with λ_n of your own choosing.

298. Prove the following theorem:

$$\sum_{n=1}^{\infty} a_n e^{-(\log \lambda_n) z} = \frac{1}{\Gamma(z)}\int_0^{\infty} s^{z-1}\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}\right) ds$$

for $x > 0$, if the series in question converge.

299. A much-studied function known as the Riemann ζ-function can be defined as follows: when $x > 1$,

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z},$$

and when $x > 0$,

$$(1-2^{1-z})\zeta(z) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-z}.$$

Show the equivalence of these two definitions when $x > 1$. Use the result of the previous exercise to express $\zeta(z)$ as an integral. Use each definition independently.

300. Express at least two additional series from Exercises 279–294 as integrals.

CHAPTER XIII

BINOMIAL COEFFICIENT SERIES

Definition 36. *A series of the type*

$$(1) \qquad \sum_{n=1}^{\infty} c_n \frac{(z-1)(z-2)\dots(z-n)}{n!}$$

is called a binomial coefficient series in z.

Theorem 180. (i) *Series* (1) *and*

$$(2) \qquad \sum_{n=1}^{\infty}\frac{(-1)^n c_n}{n^z}$$

converge and diverge at the same points with the possible exception of 1, 2, 3, ...; (ii) *if* (2) *converges uniformly over a finite region S, then so does* (1); (iii) *if* (1) *converges uniformly over a finite region S′, the points* 1, 2, 3, ... *being neither within nor on its boundary, then so does* (2); (iv) *if one series converges absolutely at any point excepting possibly* 1, 2, 3, ..., *so does the other.*

PROOF: Restrict z to S and assume (2) uniformly convergent over S. Let

$$b_n = \frac{(-1)^n(z-1)(z-2)\dots(z-n)}{n!\,n^{-z}}$$

and $a_n = (-1)^n c_n n^{-z}$. Then

$$b_n - b_{n+1} = b_n\left[1 + \frac{(z-n-1)\left(1+\frac{1}{n}\right)^z}{n+1}\right]$$

$$= b_n\left[1 + \left(-1+\frac{z-1}{n}\right)\left(1+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z\right].$$

Apply Taylor's formula to $\left(1+\frac{1}{n}\right)^{-1}$ and $\left(1+\frac{1}{n}\right)^z$ assuming $n \geqq 2$. Then

$$b_n - b_{n+1}$$
$$= b_n\left[1 + \left(-1 + \frac{z-1}{n}\right)\left(1 - \frac{1}{n} + \frac{\beta}{n^2}\right)\left(1 + \frac{z}{n} + \frac{\alpha}{n^2}\right)\right],$$

where β and α remain finite. Multiplying out,

$$b_n - b_{n+1} = b_n\left[\frac{\gamma}{n^2}\right],$$

where γ remains finite. Moreover, b_n remains finite; for, in case $z = 1, 2, 3, \ldots, b_n \to 0$, and in other cases

$$b_n \to \frac{-z}{\Gamma(-z)}.*$$

Consequently $\sum_{n=0}^{\infty} |b_n - b_{n+1}|$ converges uniformly over S. See Theorem **113**. Now apply Theorem **119** and we have (ii) of the present theorem; (iii) can be proved in an exactly analogous manner; (ii) and (iii) together include (i); (iv) is a result of the fact that b_n remains finite. See Theorem 38.

Certain results previously obtained for ordinary Dirichlet series can now be immediately inferred for binomial coefficient series. See for example Theorems **164**.

EXERCISE

301. Given that (1) converges when $z = z_0$, prove that it converges uniformly over a sector defined by

$$|am\,(z - z_0)| \leqq \pi - \delta, \quad \delta > 0.$$

* See, for example, Pierpont, *Functions of a Complex Variable*, p. 298.

CHAPTER XIV

FACTORIAL SERIES

Definition 37. *A series of the type*

$$\sum_{n=0}^{\infty} \frac{n!\,a_n}{z(z+1)\ldots(z+n)} \tag{1}$$

is called a factorial series.

The a_n's are independent of z which may be any complex number whatever except $0, -1, -2, \ldots$

Theorem 181. HYPOTHESIS: *Series* (1) *converges when* $z = z_0$. CONCLUSION: *It converges uniformly over the region defined by*

$$|am\,(z - z_0)| \leqq \omega = \pi - \delta, \quad \delta > 0,$$

from which neighbourhoods of all points $0, -1, -2, \ldots$ *which fall within it have been removed.*

PROOF: Denote the real part of z by $R(z)$ and assume $R(z_0) > 0$. There is no loss of generality in this. For if the contrary is true, suppose $R(z_0 + k) > 0$. Drop from consideration the first k terms of the series and factor

$$\frac{(k-1)!}{z(z+1)\ldots(z+k-1)}$$

from each term that remains. The removal of these terms and this factor will not affect uniform convergence over the region in question. Replace $z_0 + k$ by z_0. Make necessary changes in the coefficients.

Apply the following formula:

$$\sum_{\nu=m}^{N} u(\nu)\,\Delta v(\nu) = u(\nu)\,v(\nu)\Big]_m^{n+1} - \sum_{\nu=m}^{n} v(\nu+1)\,\Delta u(\nu).$$

Let
$$u(\nu) = -\frac{z_0(z_0+1)\dots(z_0+\nu)}{z(z+1)\dots(z+\nu)},$$

$$\Delta v(\nu) = -a_\nu \frac{\nu!}{z_0(z_0+1)\dots(z_0+\nu)},$$

and
$$v(\nu) = \sum_{n=\nu}^{\infty} a_n \frac{n!}{z_0(z_0+1)\dots(z_0+n)}.$$

Whereupon

$$(2)\quad \sum_{\nu=m}^{\infty} a_\nu \frac{\nu!}{z(z+1)\dots(z+\nu)} = \left[\frac{z_0(z_0+1)\dots(z_0+m)}{z(z+1)\dots(z+m)}\right]^* \left[\sum_{n=m}^{\infty} a_n \frac{n!}{z_0(z_0+1)\dots(z_0+n)}\right] + \sum_{\nu=m}^{\infty}(z_0-z)\cdot \left(\sum_{n=\nu+1}^{\infty} a_n \frac{n!}{z_0(z_0+1)\dots(z_0+n)}\right)\left(\frac{z_0(z_0+1)\dots(z_0+\nu)}{z(z+1)\dots(z+\nu)(z+\nu+1)}\right).$$

Let $z = x+yi$ and notice that

$$\left|\frac{z_0+n}{z+n}\right| \leqq \frac{|z_0+n|}{x+n}, \quad \frac{x+n+1}{|z+n+1|} \leqq 1,$$

and that $|z-z_0| \leqq (x-x_0)\sec\omega$. Take an $\eta > 0$ and choose m so large that

$$\left|\sum_{\nu=m}^{\infty} a_n \frac{n!}{z_0(z_0+1)\dots(z_0+n)}\right| < \eta.$$

Then, from (2),

$$(3)\quad \left|\sum_{\nu=m}^{\infty} a_n \frac{\nu!}{z(z+1)\dots(z+\nu)}\right| \leqq \eta \left|\frac{z_0(z_0+1)\dots(z_0+m)}{z(z+1)\dots(z+m)}\right| + \eta\sec\omega \sum_{\nu=m}^{\infty}(x-x_0)\frac{|z_0(z_0+1)\dots(z_0+\nu)|}{x(x+1)\dots(x+\nu)(x+\nu+1)}.$$

* We are here assuming, as is proved in a few lines, that
$$\frac{z_0(z_0+1)\dots(z_0+m)}{z(z+1)\dots(z+m)}$$
remains finite.

Next

$$\frac{|z_0(z_0+1)\dots(z_0+\nu)|}{x_0(x_0+1)\dots(x_0+\nu)} = \left(\left[1+\frac{y_0^2}{x_0^2}\right]\left[1+\frac{y_0^2}{(x_0+1)^2}\right]\dots\left[1+\frac{y_0^2}{(x_0+\nu)^2}\right]^*\right)^{\frac{1}{2}} < M,$$

a constant. Moreover,

$$\left|\frac{z_0(z_0+1)\dots(z_0+m)}{z(z+1)\dots(z+m)}\right| = \frac{|z_0(z_0+1)\dots(z_0+m)|}{x_0(x_0+1)\dots(x_0+m)}\cdot\frac{x_0(x_0+1)\dots(x_0+m)}{|z(z+1)\dots(z+m)|} < M.$$

Make these substitutions in (3).

$$\left|\sum_{\nu=m}^{\infty} a_\nu \frac{\nu!}{z(z+1)\dots(z+\nu)}\right| \leqq M\eta + M\eta\sec\omega\sum_{\nu=m}^{\infty}\Delta\frac{x_0(x_0+1)\dots(x_0+\nu)}{x(x+1)\dots(x+\nu)}$$
$$= M\eta + M\eta\sec\omega\frac{x_0(x_0+1)\dots(x_0+m)}{x(x+1)\dots(x+m)} < M\eta(1+\sec\omega),$$

which can be made as small as desired by a proper choice of η. Uniform convergence follows.

Theorem 182. $\sum_{n=0}^{\infty}\frac{n!\,a_n}{z(z+1)\dots(z+n)}$ *either* (i) *converges for all values of* z *except* $0, -1, -2, \dots$, *or* (ii) *diverges for all such values of* z, *or* (iii) *there exists a line,* $x = x_0$, *such that when* $x > x_0$ *and* $z \neq 0, -1, -2, \dots$, *the series converges, and when* $x < x_0$ *and* $z \neq 0, -1, -2, \dots$, *diverges.*

The possibilities (i) and (ii) are shown readily by letting $a_n = \left(\frac{1}{n!}\right)^2$ and $(n!)^2$ respectively, and considering the resulting special series. (iii) is a result of Theorem 181 coupled with an example of a factorial series which converges at one point and diverges at another as in the corresponding theorem

* It can be proved that a product $(1+a_0)(1+a_1)\dots(1+a_n)$ remains finite when $n\to\infty$ if $\sum_{n=0}^{\infty} a_n$ converges.

for Dirichlet series. Such an example is

$$\sum_{n=0}^{\infty} \frac{n!}{z(z+1)\dots(z+n)}.$$

Here $$\lim_{n\to\infty} n\left[\frac{c_n}{c_{n+1}} - 1\right] = z-1,$$

where c_n is the general term of the series. See Theorem 42.

Factorial series have a theory closely paralleling that of Dirichlet series. The similarity between Theorems 163 and 181 is to be noted. As a matter of fact ordinary Dirichlet series can be exhibited as the limiting case of a class of series which have factorial series as a special case. Common properties of factorial series and ordinary Dirichlet series then appear simply as properties of series of the general type. A class of series of this character will be studied in the next chapter, and, as a consequence, the study of factorial series is not carried farther at the present time.

EXERCISES

302. Prove that if (1) converges when $z = z_0$, it converges uniformly over the half-plane defined by $R(z) \geqq R(z_0) + \delta$, $\delta > 0$, from which half-plane neighbourhoods of those points $0, -1, -2, \dots$ which lie in it have been removed.

303. Prove the theorem. HYPOTHESIS: Series (1) converges absolutely when $z = z_0$. CONCLUSION: It converges absolutely uniformly when $R(z) \geqq R(z_0)$.

304. Factorial series can be generalized by replacing n of series (1) by λ_n, where $\lambda_n < \lambda_{n+1} \to \infty$. Prove Theorem 181 for this more general series.

305. Replace a_n of series (1) by $a_n(z)$. State and prove a theorem relative to the resulting series.

CHAPTER XV

GENERALIZED FACTORIAL SERIES

Consider the series

$$(1)\qquad \sum_{n=1}^{\infty} a_n A_n^{(k)}(z),$$

where $$A_n^{(k)}(z) = \frac{k}{z+k-1}\,\frac{\Gamma(nk)}{\Gamma(z+nk)}\,\frac{\Gamma(z+k)}{\Gamma(k)}$$

whenever z, k, and n have such values that this formula defines a number. Whenever, for a particular point (z_0, k_0, n_0), $A_n^{(k)}(z)$ is not defined by the formula but approaches a limit as (z, k, n) approaches (z_0, k_0, n_0), $A_n^{(k)}(z)$ is given at this point this limiting value.

Definition 38. *Series of type* (1) *will be called generalized factorial series.*

Theorem 183. *When* $k=1$, (1) *reduces to the ordinary factorial series*

$$(2)\qquad \sum_{n=1}^{\infty} a_n \frac{(n-1)!}{z(z+1)\dots(z+n-1)}.$$

Proof is a mere matter of substitution, remembering that $\Gamma(n) = (n-1)!$.

Before giving the next fundamental theorem we insert a lemma.

Theorem 184 (*Lemma*). HYPOTHESIS: $\delta > 0$, $|q+z| \geqq \delta$, $|q| \geqq \delta$, $|am\,(q)| \leqq \pi - \delta$, $|am\,(q+z)| \leqq \pi - \delta$. CONCLUSION:

$$(3)\qquad \frac{q^z\,\Gamma(q)}{\Gamma(q+z)} = 1 + \frac{z-z^2}{2q} + \frac{\psi(z,q)}{q^2},$$

where $|\psi| < M(z)$ *independent of* q.

PROOF: Consider the asymptotic* form for $\Gamma(s)$,

$$\Gamma(s) = e^{(s-\frac{1}{2})\log s - s + \log\sqrt{2\pi} + \omega(s)},$$

* See, for example, Pierpont, *Functions of a Complex Variable*, p. 327.

where $\omega(s) = \frac{c_1}{s} + \frac{f_1(s)}{s^2}$, c_1 being a constant and $|f_1(s)| < c_2$, a constant; which asymptotic form is valid when $|s| \geqq$ [illegible] > 0 and $|am(s)| \leqq \pi - \epsilon$. Substitute in the left-hand member of (3)

$$\frac{q^z\,\Gamma(q)}{\Gamma(q+z)} = \left(1+\frac{z}{q}\right)^{\frac{1}{2}-z} {}^{-q\log\left(1+\frac{z}{q}\right)-z}\, e^{\omega(q)-\omega(q+z)}.$$

Assume $|q| > |z| + \delta$ and expand

$$\left(1+\frac{z}{q}\right)^{\frac{1}{2}-z} \quad \text{and} \quad \log\left(1+\frac{z}{q}\right)$$

as functions of $\frac{z}{q}$ by the analogue for complex variables of the law of the mean *

$$\frac{q^z\,\Gamma(q)}{\Gamma(q+z)} = \left[1+\left(\frac{1}{2}-z\right)\frac{z}{q}+\frac{\alpha_1}{q^2}\right]\left[e^{\frac{z^2}{2q}+\frac{\alpha_2}{q^2}}\right].$$

$$\left[e^{\omega(q)-\omega(q+z)}\right] = \left[1+\left(\frac{1}{2}-z\right)\frac{z}{q}+\frac{\alpha_1}{q^2}\right]\left[\frac{z^2}{2q}+\frac{\alpha_3}{q^2}\right]\left[1+\frac{\alpha_4}{q^2}\right],$$

where $|\alpha_1|, |\alpha_3|, |\alpha_4|$ are all less than $\bar{M}$ (independent of q). Multiplying out we have the desired formula.

If $|q| \leqq |z| + \delta$, as all singularities are excluded, we simply solve (3) for ψ. It is determinable as a finite function.

Theorem 185. *When $k \to \infty$, in such a way that*

$$|am(k)| \leqq \pi - \epsilon,$$

$\epsilon > 0$, (1) *reduces to the ordinary Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}.$$

PROOF: We write

$$(4) \qquad A_n^{(k)}(z)\,n^z = \frac{k}{z+k-1}\,\frac{\Gamma(nk)\,(nk)^z}{\Gamma(z+nk)}\cdot\frac{\Gamma(z+k)}{\Gamma(k)\,k^z}.$$

Substitute from (3) and the theorem is immediate.

We shall understand $A_n^{(\infty)}(z) \equiv \frac{1}{n^z}$; and when the symbol

* See, for example, W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. i, p. 316, 2nd ed.

$A_n^{(k)}(z)$ is used and no restriction placed on k we shall understand $A_n^{(\infty)}(z) \equiv \frac{1}{n^z}$ to be included.

Theorem 186. HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n A_n^{(k_0)}(z)$ *converges*; (ii) $k \neq 0$, $|am(k)| \leqq \pi - \epsilon$ *and* $|z+k+j| \geqq \epsilon$, $j = 0, 1, 2, \ldots$, $\epsilon > 0$. CONCLUSION: $\sum_{n=1}^{\infty} a_n A_n^{(k)}(z)$ *converges.*

PROOF: Consider $A_n^{(k)}(z)\,n^z$ as given by (4). Take n so large, say greater than n_0, that $|am(nk+z)| < \pi - \epsilon$, then

$$\frac{(nk)^z\,\Gamma(nk)}{\Gamma(nk+z)} = 1 + \frac{z-z^2}{2\,nk} + \frac{\psi(z, nk)}{n^2k^2},$$

where $|\psi| < M_2$, independent of n and k. Moreover, again by (3) $\frac{\Gamma(z+k)}{\Gamma(k)\,k^z} \to 1$ as $k \to \infty$, and since all its singularities are excluded from the region in question and $\Gamma(z+k) > \delta > 0$ in this region, it remains finite and in absolute value greater than a fixed ϵ. Consequently, when $n > n_0$,

$$(5) \qquad A_n^{(k)}(z)\,.\,n^z = f_2(z,k) + \frac{f_3(z,k)}{n} + \frac{f_4(z,k,n)}{n^2},$$

where $|f_2|, |f_3|, |f_4| < M$, a constant. It can readily be shown that $\frac{1}{A_n^{(k)}(z)}$ is of the same form and consequently

$$A_n^{(k)}(z) \,/\, A_n^{(k_0)}(z)$$

also, even if $A_n^{(k_0)}(z) \neq \frac{1}{n^z}$. If desired, any convenient definition can be given to f_2, f_3, and f_4 for values of $n < n_0$ so that (5) will hold for all values of n.

The theorem now follows by Theorems 26, 27, and 38.

Corollary. *A common line of convergence for* (1), *for all values of k considered, follows immediately. See Theorems* 164 *and* 182.

Theorem 187. HYPOTHESIS: $\sum_{n=1}^{\infty} a_n A_n^{(k_0)}(z)$ *converges*

absolutely. CONCLUSION: $\sum_{n=1}^{\infty} a_n A_n^{(k)}(z)$ *converges absolutely also, k restricted as in the previous theorem.*

Proof is as for the last theorem and is omitted.

Corollary. *A common line of absolute convergence for* (1), *for all values of k considered in the theorem, follows.*

We denote the common line of convergence by $x = \alpha$ and of absolute convergence by $x = \beta$. We know by Theorem 169 that $\beta - \alpha \leqq 1$.

Theorem 188. HYPOTHESIS: *Series* (1) *converges when* $z = z_0 = x_0 + y_0 i$, $k = k_0$. CONCLUSION: (1) *converges uniformly in z and k over the region S, defined by the following inequalities*: $|z| < M$, $|am(z - z_0)| \leqq \frac{\pi}{2} - \delta$, $|am(k)| \leqq \pi - \delta$, $k > \delta$, $|z + k + j| > \delta$. *M and δ are any positive constants, so long as* $\delta < \frac{\pi}{2}$, $j = 0, 1, 2, \ldots$.

PROOF: By formula (5)

$$\sum_{n=1}^{\infty} a_n A_n^{(k)} = f_2 \sum_{n=1}^{\infty} a_n n^{-z} + f_3 \sum_{n=1}^{\infty} a_n n^{-z} \cdot \frac{1}{n} + \sum_{n=1}^{\infty} a_n n^{-z} \cdot \frac{f_4}{n^2},$$

where $|f_2|$, $|f_3|$, $|f_4|$ remain finite over S. We know from Theorem 163 that $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly over the region defined by $|am(z - z_0)| \leqq \frac{\pi}{2} - \delta$. Of the series in the right-hand member the first is seen immediately to converge uniformly over S, since $\sum_{n=1}^{\infty} a_n n^{-z}$ does, and since f_2 remains finite. The second does the same. To prove it apply Theorem 119, letting a_n of that theorem be $a_n n^{-z}$ and $b_n = \frac{1}{n}$. Next, since $|f_4| < M$, the last series is proved uniformly convergent by comparison with $\sum_{n=1}^{\infty} \frac{M}{n^2}$. (See Theorem 113.)

Since the series formed by adding like terms of three uniformly convergent series is uniformly convergent, our theorem follows.

If we give k a fixed real positive value we do not find it necessary to limit z to a finite domain. We have the following theorem which extends the domain of z to the right of the line of absolute convergence. We recall also the fact that the difference between the abscissas of absolute convergence and of convergence is at most 1.

Theorem 189. HYPOTHESES: (i) *Series* (1) *converges absolutely when* $z = z_0$ *and* $k = k_0 > 0$ *simultaneously*; (ii) $R(z_0) = x_0 > 0$. CONCLUSION: *Series* (1) *converges absolutely and uniformly in z over the half plane* $x \geqq x_0$ *from which fixed neighbourhoods of those points*, $-(k_0 - 1)$, $-k_0$, $-(k_0 + 1)$, ..., *which occur in it, have been removed.*

PROOF: We begin by letting $z = x$ be real, positive, and at least as large as x_0. Then $\frac{k_0}{x + k_0 + 1}$ decreases as x increases. The same is true of $\frac{\Gamma(x+k)}{\Gamma(x+nk)}$. To prove this last statement, consider the infinite product*

$$\frac{\Gamma(x+k)}{\Gamma(x+nk)} = \frac{x+nk}{x+k} e^{c(nk-k)} \prod_{N=1}^{\infty} \frac{1 + \frac{x+nk}{N}}{1 + \frac{x+k}{N}} e^{\frac{k-nk}{N}}.$$

When $n > 1$, $\frac{x+nk}{x+k}$ decreases as x increases. This is readily proved by differentiation. The same is true of

$$\frac{1 + \frac{x+nk}{N}}{1 + \frac{x+k}{N}}.$$

Hence, convergence of (1) is uniform and absolute when z is real and $\geqq x_0$ (see Theorem 113). We can prove in like

* See, for example, Pierpont, *Theory of Functions of a Complex Variable*, p. 298.

manner that $\left|\frac{\Gamma(z+k)}{\Gamma(z+nk)}\right|$ decreases as $|y|$ increases, $x > 0$ being held fixed. The theorem follows.

EXERCISE

306. In the proof of Theorem 189 it was assumed that $R(z_0) > 0$. Prove this restriction unnecessary.

CHAPTER XVI

FOURIER SERIES

§ 1. Definition.

The topic of Fourier series is very briefly treated in this volume. The purpose is simply to give some of the fundamental definitions and theorems with such brevity and simplicity as is possible.

Throughout the chapter we shall deal only with real numbers.

Definition 39. *A series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \tag{2}$$

is called a Fourier series.

It is said to correspond to $f(x)$.

The definition presupposes the existence of the integrals* defining the coefficients. It makes no assumption relative to convergence of the series.

A transformation of variable $x = \gamma x' + \delta$ will replace $\sin nx$ and $\cos nx$ by $\sin n(\gamma x' + \delta)$ and $\cos n(\gamma x' + \delta)$ respectively, and the interval $(-\pi, \pi)$ by (α, β), where

$$\alpha = -\frac{1}{\gamma}(\pi + \delta) \quad \text{and} \quad \beta = \frac{1}{\gamma}(\pi - \delta).$$

This does not essentially alter the character of the series.

* Modern studies on Fourier series usually use the Lebesgue definition of an integral. However, in order to make the chapter elementary and brief the Riemann definition is used in the subsequent discussion. This sacrifices much in generality.

For an extended study consult such a book as *The Theory of Functions of a Real Variable*, by E. W. Hobson.

Definition 40. *a_n and b_n, $n = 1, 2, \ldots$, are called the Fourier constants for $f(x)$.*

The following are examples of Fourier series. It is to be observed that no statement is made here relative to convergence, or in the case of convergence to the function represented.

I. $\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \ldots .$

Here, $f(x) = \begin{cases} -\frac{1}{2}(x+\pi) & \text{when } x \leqq 0, \\ -\frac{1}{2}(x-\pi) & \text{when } x > 0. \end{cases}$

II. $\frac{8}{\pi}\left[\frac{1}{2} \sin 2x + \frac{1}{6} \sin 6x + \frac{1}{10} \sin 10x + \ldots\right].$

Here, $f(x) = \begin{cases} 1 & \text{when } -\pi \leqq x \leqq -\frac{1}{2}\pi,\ 0 \leqq x \leqq \frac{1}{2}\pi, \\ -1 & \text{when } -\frac{1}{2}\pi < x < 0,\ \frac{1}{2}\pi < x < \pi. \end{cases}$

III. $\frac{4}{\pi}\left[\sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \ldots\right].$

$$\text{Here, } f(x) = \begin{cases} -\pi - x & \text{when } -\pi \leqq x < -\frac{\pi}{2}, \\ x & \text{when } -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}, \\ \pi - x & \text{when } \frac{\pi}{2} < x \leqq \pi. \end{cases}$$

IV. $$\frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2} \cos x + \frac{1}{1+1^2} \sin x + \frac{1}{1+2^2} \cos 2x - \frac{2}{1+2^2} \sin 2x - \frac{1}{1+3^2} \cos 3x + \frac{3}{1+3^2} \sin 3x - \ldots\right].$$

Here, $f(x) = e^x$ when $-\pi < x \leqq \pi$.

We observe the following theorems.

Theorem 190. *If a Fourier series converges at every point of an interval of length 2π closed at one end, it converges everywhere and to a function with period 2π.*

Proof is immediate.

Theorem 191. *If $f(x)$ is an even function the corresponding Fourier series will contain only cosine terms, if it is an odd function, only sine terms.*

Proof is also immediate.

§ 2. Fundamental theorems.*

The Fourier series corresponding to many functions, $f(x)$, converge to the value $f(x)$. This is most important as the Fourier series is then a representation of the function in terms of elementary functions. Unfortunately the question as to when Fourier series converge and to what value is not easily answered. We proceed here to study the question for an important class of functions.

Let

$$\begin{aligned} S_n &= \frac{a_0}{2} + \sum_{n=1}^{n} (a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha \\ &+ \sum_{n=1}^{n} \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(\alpha) \cos nx \cos n\alpha\, d\alpha + \int_{-\pi}^{\pi} f(\alpha) \sin nx \sin n\alpha\, d\alpha\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\sum_{n=1}^{n}\int_{-\pi}^{\pi} f(\alpha) \cos n(x-\alpha)\, d\alpha. \end{aligned}$$

We now assume the trigonometric formula,

$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{\sin \frac{1}{2}(2n+1)\theta}{2 \sin \frac{1}{2}\theta}, \quad \theta \neq 0.$$

This gives us

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \frac{\sin \frac{1}{2}(2n+1)(x-\alpha)}{\sin \frac{1}{2}(x-\alpha)}\, d\alpha.$$

Change the variable by letting $\alpha = x + 2y$ and let $2n+1 = m$. We have

$$(1) \qquad S_n(x) = \frac{1}{\pi}\int_{-\frac{1}{2}(\pi+x)}^{\frac{1}{2}(\pi-x)} f(x+2y) \frac{\sin my}{\sin y}\, dy.$$

This is known as the Dirichlet integral, although the term is also applied to more general forms. We notice that this integral is extended over an interval of length π. For purposes of compactness in writing we proceed by making $f(x)$, which until the present has been defined only over the interval

* Theorem 217 of chap. XVII is just as fundamental as any theorem of this chapter.

$(-\pi, \pi)$, a periodic function with period 2π. By this rule $f(x \pm 2\pi) \equiv f(x)$. If for a particular value of x, $f(x)$ is not defined, the same is to be true of $f(x \pm 2\pi)$. For consistency, if necessary, we re-define $f(-\pi)$ so that $f(-\pi) = f(\pi)$. It results that $f(x+2y)$ has the period π in y. Since m is an odd integer, $\sin my / \sin y$ has this period also. As a consequence, the value of the integral in (1) will not be affected if the limits of integration are changed, so long as the interval of integration remains of length π. Hence, replacing $f(x+2y)$ by $F(y)$, we write

$$(2) \quad S_n(x) = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F(y)\frac{\sin my}{\sin y}\,dy$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} F(y)\frac{\sin my}{\sin y}\,dy + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} F(-y)\frac{\sin my}{\sin y}\,dy.$$

Before proceeding with Theorem 194 which is a fundamental theorem we prove two lemmas.

Theorem 192 (*Lemma* 1). HYPOTHESIS:

$$0 < \alpha < \beta \leqq \frac{\pi}{2}.$$

CONCLUSION:

$$\lim_{m\to\infty}\int_\alpha^\beta \frac{\sin my}{\sin y}\,dy = 0.$$

PROOF: Since $\dfrac{1}{\sin y}$ is a monotonic function over the interval in question, by the second mean value theorem *

$$\int_\alpha^\beta \frac{\sin my}{\sin y}\,dy = \frac{1}{\sin\alpha}\int_\alpha^\gamma \sin my\,dy + \frac{1}{\sin\beta}\int_\gamma^\beta \sin my\,dy,$$

where $\alpha \leqq \gamma \leqq \beta$.

* If $f(x)$ is bounded and monotonic and hence integrable over the interval (a, b), and if $\phi(x)$ is bounded and integrable over the same interval,

$$\int_a^b f(x)\,\phi(x)\,dx = f(a+0)\int_a^\xi \phi(x)\,dx + f(b-0)\int_\xi^b \phi(x)\,dx,\ a \leqq \xi \leqq b.$$

See, for example, *Functions of a Real Variable*, by E. W. Hobson, p. 362, 1907 ed.

Perform the integration and we readily get

$$(3) \qquad \left|\int_\alpha^\beta \frac{\sin my}{\sin y}\,dy\right| < \frac{2}{m}(\csc\alpha + \csc\beta) < \frac{4}{m}\csc\alpha,$$

which approaches zero when $m \to \infty$.

Theorem 193 (*Lemma* 2). HYPOTHESIS: $0 \leqq \alpha < \beta$. CONCLUSION:

$$(4) \qquad \left|\int_\alpha^\beta \frac{\sin x}{x}\,dx\right| < K,$$

independent of α and β.

PROOF: Denote by $\bar{n}$ that integer such that

$$\bar{n}\pi \leqq \beta < (\bar{n}+1)\pi.$$

$$\int_0^\beta \frac{\sin x}{x}\,dx = \int_0^\pi \frac{\sin x}{x}\,dx + \int_\pi^{2\pi} \frac{\sin x}{x}\,dx + \ldots + \int_{\bar{n}\pi}^\beta \frac{\sin x}{x}\,dx.$$

In case $\beta < \pi$, $\bar{n} = 0$, and the last integral is the only one to appear. Let $\beta \to \infty$. We get in the right-hand member a convergent infinite series (see Theorem 60). In other words

$$\lim_{\beta\to\infty}\int_0^\beta \frac{\sin x}{x}\,dx$$

exists. Consequently there exists a number, H, such that

$$\left|\int_0^\beta \frac{\sin x}{x}\,dx\right| < H$$

always. Hence

$$\left|\int_\alpha^\beta \frac{\sin x}{x}\,dx\right| = \left|\int_0^\beta \frac{\sin x}{x}\,dx - \int_0^\alpha \frac{\sin x}{x}\,dx\right|$$

$$\leqq \left|\int_0^\beta \frac{\sin x}{x}\,dx\right| + \left|\int_0^\alpha \frac{\sin x}{x}\,dx\right| < 2H = K.$$

Theorem 194. HYPOTHESIS: *$F(y)$ is bounded and does not decrease as y increases in the interval $0 < y < \frac{\pi}{2}$.* CONCLUSION:

$$\lim_{m\to\infty}\int_0^{\frac{\pi}{2}} F(y)\frac{\sin my}{\sin y}\,dy = \frac{\pi}{2}F(+0).$$

PROOF: The existence of $F(+0) = \lim_{y \to 0} F(y)$, $y > 0$, follows from the hypothesis (see Theorem 18). Moreover,

$$\int_0^{\frac{\pi}{2}} F(y) \frac{\sin my}{\sin y}\, dy$$
$$= F(+0) \int_0^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy + \int_0^{\mu} [F(y) - F(+0)] \frac{\sin my}{\sin y}\, dy$$
$$+ \int_{\mu}^{\frac{\pi}{2}} [F(y) - F(+0)] \frac{\sin my}{\sin y}\, dy,$$

where μ is any number between 0 and $\frac{\pi}{2}$. Let

$$G(y) = [F(y) - F(+0)] \frac{y}{\sin y}.$$

Note that

$$\int_0^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy = \int_0^{\frac{\pi}{2}} (1 + 2\cos 2y + \ldots + 2\cos 2ny)\, dy = \frac{\pi}{2},$$

then apply the second mean value theorem to

$$\int_{\mu}^{\frac{\pi}{2}} [F(y) - F(+0)] \frac{\sin my}{\sin y}\, dy$$

and we have

$$(5) \quad \int_0^{\frac{\pi}{2}} F(y) \frac{\sin my}{\sin y}\, dy - \frac{\pi}{2} F(+0)$$
$$= \int_0^{\frac{\pi}{2}} F(y) \frac{\sin my}{\sin y}\, dy - F(+0) \int_0^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy$$
$$= \int_0^{\mu} G(y) \frac{\sin my}{y}\, dy + [F(\mu + 0) - F(+0)] \int_{\mu}^{\xi_1} \frac{\sin my}{\sin y}\, dy$$
$$+ \left[F\left(\frac{\pi}{2} - 0\right) - F(+0)\right] \int_{\xi_1}^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy.$$

Now $F(y)$ does not decrease as y increases from 0 to μ, $\frac{y}{\sin y}$ increases, $0 < y \leqq \mu$, as is readily proved by differentiation. Hence $G(y)$ does not decrease as y increases in the interval $0 < y \leqq \mu$ and approaches a limit $G(\mu - 0)$ when $y \to \mu$ from below. Consequently, again by the second mean value theorem,

$$\int_0^{\mu} G(y) \frac{\sin my}{y}\, dy = G(\mu - 0) \int_{\xi}^{\mu} \frac{\sin my}{y}\, dy$$
$$= G(\mu - 0) \int_{m\xi}^{m\mu} \frac{\sin y}{y}\, dy,$$

where $0 \leqq \xi \leqq \mu$.

Hence, by lemma 2,

$$(6) \qquad \left| \int_0^{\mu} G(y) \frac{\sin my}{y}\, dy \right| < K G(\mu).$$

But μ can be taken as close to zero as desired, in particular, inasmuch as $G(\mu - 0) \to 0$ when $\mu \to 0$, so close that

$$|G(\mu - 0)| < \frac{\epsilon}{2K}.$$

Moreover, by lemma 1,

$$\int_{\mu}^{\xi_1} \frac{\sin my}{\sin y}\, dy \quad \text{and} \quad \int_{\xi_1}^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy$$

both approach zero when $m \to \infty$. Take m_1 so large, that whenever $m > m_1$,

$$\left| [F(\mu + 0) - F(+0)] \int_{\mu}^{\xi_1} \frac{\sin my}{\sin y}\, dy \right.$$
$$\left. + \left[F\left(\frac{\pi}{2} - 0\right) - F(+0)\right] \int_{\xi_1}^{\frac{\pi}{2}} \frac{\sin my}{\sin y}\, dy \right| < \frac{\epsilon}{2}.$$

Then,

$$\left| \int_0^{\frac{\pi}{2}} F(y) \frac{\sin my}{\sin y}\, dy - \frac{\pi}{2} F(+0) \right| < \epsilon,$$

and our theorem is proved.

This theorem holds for a function $F(y)$ which is bounded and never increases; for apply it, as proved, to $-F(y)$ which never decreases.

If we apply the above theorem coupled with a similar result for the second integral in the third member of (2) and remember that $F(y) = f(x + 2y)$ we have the following theorem where $f(x \pm 0) = \lim_{x \to 0} f(x)$, $x \gtrless 0$.

Theorem 195. HYPOTHESIS: *$f(x)$ is bounded and monotonic over the interval $(-\pi, \pi)$.* CONCLUSION: *The corresponding Fourier series, that is, series (1) of § 1, converges to*

$$\tfrac{1}{2}[f(x + 0) + f(x - 0)].$$

Now any function of bounded variation on an interval can be represented as the sum of two bounded functions monotonic on that interval.* This fact gives us the following more general theorem.

Theorem 196. HYPOTHESIS: *$f(x)$ is of bounded variation on the interval* $(-\pi, \pi)$. CONCLUSION: *Series* (1) *of* § 1 *converges to* $\frac{1}{2}[f(x+0)+f(x-0)]$.

Corollary. *Under the hypothesis of the theorem, if* $x=\pi$ *or* $-\pi$, (1) *of* § 1 *converges to* $\frac{1}{2}[f(\pi-0)+f(-\pi+0)]$.

To prove the corollary, as already explained, define $f(x)$ at the end points and outside the interval $(-\pi, \pi)$ so as to make it periodic with period 2π. Then, on account of the periodic character of all functions that enter, the interval $(-\pi, \pi)$ can be replaced by any interval of length 2π without in any way affecting our work and the point π can be made an interior point.

The proof of Theorem 196 depended among other things upon the fact that

$$\lim_{m\to\infty}\int_{\mu}^{\frac{\pi}{2}}[F(y)-F(+0)]\frac{\sin my}{\sin y}dy=0,$$

where μ is any number so long as $0<\mu\leqq\frac{\pi}{2}$. This was proved by applying the second mean value theorem to the integral which in turn demanded the condition that $F(y)$ be monotonic and hence that $f(x)$ be of bounded variation. The integral in question can be proved † to approach 0, however, by other methods which do not make this demand. If we grant this, remembering that $\mu>0$ is as small as desired, we have the following theorem.

Theorem 197. HYPOTHESIS: *There exists a neighbourhood surrounding x in which $f(x)$ is of bounded variation.* CONCLUSION: *Series* (1) *of* § 1 *converges to* $\frac{1}{2}[f(x+0)+f(x-0)]$.

We next remark without proof the following interesting facts relative to Fourier series.

* See, for example, Hobson, *Theory of Functions of a Real Variable*, vol. ii, p. 703, ed. 1926.

† See, for example, Hobson, *Theory of Functions of a Real Variable*, p. 679, ed. 1907.

The condition 'bounded variation' that has been put on $f(x)$ is sufficient but not necessary for convergence.

There exist functions which are continuous at all points of the interval $-\pi\leqq x\leqq\pi$ but whose corresponding Fourier series do not converge at all points.

In other words, continuity is not a sufficient condition for convergence.

§ 3. Uniform convergence of Fourier series.

Theorem 198. HYPOTHESIS: *$f(x)$ is of bounded variation on any finite interval and is periodic with period 2π.* CONCLUSION: *The corresponding Fourier series, that is series* (1) *of* § 1, *converges uniformly over any interval, I, which contains neither in its interior nor at an end point any point of discontinuity of $f(x)$.*

PROOF: We shall consider x ■ variable.

As a consequence of (5) of § 2 together with (3) and (6) of that section we have for any point x of the interval, the relation

$$\left|\int_0^{\frac{\pi}{2}}F(y)\frac{\sin my}{\sin y}dy-\frac{\pi}{2}F(+0)\right|<KG(\mu)$$

$$+\frac{4}{m\sin\mu}[F(\mu+0)-F(+0)]+\frac{4}{m\sin\xi}\left[F\left(\frac{\pi}{2}-0\right)-F(+0)\right]$$

$$<KG(\mu)+\frac{4}{m\sin\mu}\left[F(\mu+0)-F(+0)+F\left(\frac{\pi}{2}-0\right)-F(+0)\right].$$

From this inequality and the corresponding one for $F(-y)$ we get

$$|S_m(x)-f(x)|<\csc\mu\cdot\left(\frac{K\mu}{\pi}+\frac{4}{m\pi}\right)\cdot$$
$$[|f(x+2\mu)-f(x)|+|f(x-2\mu)-f(x)|]$$
$$+\frac{4\csc\mu}{m\pi}[|f(x+\pi)-f(x)|+|f(x-\pi)-f(x)|]$$
$$<\frac{K\mu}{\pi}\csc\mu\,.\,[|f(x+2\mu)-f(x)|+|f(x-2\mu)-f(x)|]$$
$$+\frac{A}{m}\csc\mu,$$

where A is chosen independent of m such that $|f(x)|<A$.

Now since I is a closed interval, $f(x)$ is uniformly continuous* over I. Hence, given any ϵ, there corresponds a $\mu_1 > 0$ such that when $\mu < \mu_1$,

$$|f(x+2\mu)-f(x)| < \tfrac{1}{4}\epsilon$$

and

$$|f(x-2\mu)-f(x)| < \tfrac{1}{4}\epsilon$$

for all x's of I simultaneously. Then,

$$|S_m(x)-f(x)| < \tfrac{1}{2}\epsilon\frac{K\mu}{\pi}\csc\mu + \frac{A}{m}\csc\mu.$$

Let μ_2 be so chosen that $\dfrac{K\mu}{\pi}\csc\mu < K$ when $\mu < \mu_2$. We require that $\mu < \mu_1$ and $\mu < \mu_2$. Then,

$$|S_m(x)-f(x)| < \tfrac{1}{2}\epsilon K + \frac{A}{m}\csc\mu.$$

Hold μ fixed and take m so large that $\dfrac{A}{m}\csc\mu < \dfrac{\epsilon}{2K}$. We then have $|S_m(x)-f(x)| < \epsilon K < \eta$ if $\epsilon < \dfrac{\eta}{K}$, establishing the theorem.

Corollary. HYPOTHESES: (i) *$f(x)$ is continuous over the intervals $\gamma \leqq x \leqq \pi$ and $\delta \geqq x \geqq -\pi$*; (ii) $f(\pi) = f(-\pi)$. CONCLUSION: *The corresponding Fourier series converges uniformly over these intervals.*

If $f(x)$ is of bounded variation the corresponding Fourier series, being a series of continuous functions, cannot converge uniformly over a neighbourhood including a point of discontinuity of $f(x)$. Let x_0 be such a point. The character of the function

$$s_n(x) = \frac{a_0}{2} + \sum_{n=1}^{n}(a_n\cos nx + b_n\sin nx)$$

in the neighbourhood of x_0 is of some interest. Consider the function, $s_n(x)-f(x)$, and let x_1 be the abscissa of its extreme value nearest to x_0. It has been found that when $n\to\infty$, $x_1\to x_0$ but that $s_n(x_1)-f(x_1)$ does not approach zero.

* See, for example, Hobson, *Theory of Functions of a Real Variable*, p. 229, ed 1907.

Similar results hold for the second, third, &c., extreme values counting away from x_0. This has been called the Gibbs phenomenon and has been studied in detail. No further discussion will be given here. Somewhat similar situations are common in the theory of non-uniformly convergent series and should be already familiar to the reader.

§ 4. Differentiation and integration of Fourier series.

We refer to Theorems 122 and 124 and Theorem 198 from which important results can be drawn immediately. However, we proceed to some other general theorems.

Theorem 199. HYPOTHESES: (i) *$f(x)$ is continuous* $-\pi \leqq x \leqq \pi$; (ii) $f(-\pi) = f(\pi)$; (iii) *$f'(x)$ is bounded and integrable from $-\pi$ to π.* CONCLUSION: *The Fourier series corresponding to $f'(x)$ is obtained by term by term differentiation of the Fourier series corresponding to $f(x)$.*

Notice that no conclusion is drawn relative to the convergence of the Fourier series corresponding to $f'(x)$.

PROOF: Denote the Fourier constants for $f(x)$ by a_n and b_n and for $f'(x)$ by a'_n and b'_n. Then, when $n \geqq 1$, by integration by parts we obtain $a'_n = nb_n$ and $b'_n = -na_n$. Also

$$a'_0 = \frac{1}{\pi}[f(\pi)-f(-\pi)] = 0.$$

The theorem follows.

Theorem 200. HYPOTHESES: (i) *$f(x)$ is integrable from* $-\pi$ *to* π; (ii) $-\pi \leqq \alpha < \beta \leqq \pi$. CONCLUSION: $\displaystyle\int_\alpha^\beta f(x)\,dx$ *is represented by the convergent series obtained by term by term integration of the Fourier series corresponding to $f(x)$.*

PROOF: Let $\phi(x) = \displaystyle\int_{-\pi}^{x} f(x)\,dx$. $\phi(x)$ is continuous and of bounded variation. Hence, it can be represented by a Fourier series,

$$(1) \qquad \tfrac{1}{2}a'_0 + \sum_{n=1}^{\infty}(a'_n\cos nx + b'_n\sin nx),$$

which converges uniformly over any interval interior to

$-\pi \leq x < \pi$. If $n \geqq 1$, we readily find by integration by parts that

$$a_n' = -\frac{1}{n} b_n \text{ and } b_n' = \frac{1}{n} b_n - \frac{1}{n} a_0 \cos n\pi.$$

a_0' is determined by letting $x = -\pi$ in (1). Whereupon,

$$\tfrac{1}{2} a_0' - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos n\pi = \tfrac{1}{2}[\phi(-\pi) + \phi(\pi)] = \tfrac{1}{2}\pi a_0.$$

Moreover, when $-\pi < x < \pi$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \cos n\pi\right) \sin nx = -\tfrac{1}{2} x.$$

The substitution of these values gives

$$\phi(x) = \tfrac{1}{2} a_0 (\pi + x) + \sum_{n=1}^{\infty} \frac{1}{n} [a_n \sin nx + b_n (\cos n\pi - \cos nx)].$$

This is the series obtained by term by term integration from $-\pi$ to x. If $x = \pi$, term by term integration gives

$$\phi(\pi) = a_0 \pi$$

which is also given by this series. The theorem is now established by integrating from $-\pi$ to α and from $-\pi$ to β and subtracting.

§ 5. Uniqueness theorem.

Theorem 201 (*Lemma*). HYPOTHESES: (i) $F(x)$ *is continuous on the interval*

$$(1) \qquad \alpha \leqq x \leqq \beta;$$

$$\text{(ii)} \qquad \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \to 0,$$

when $h \to 0$, *at every point of* (1). CONCLUSION: $F(x)$ *is linear over* (1).

PROOF: Let

$$\phi(x) = \pm \left\{F(x) - F(\alpha) - \frac{x-\alpha}{\beta-\alpha}[F(\beta) - F(\alpha)]\right\} + k(x-\alpha)(x-\beta),$$

where $k > 0$ is a constant. No matter which sign we choose

for $\phi(x)$ we see that $\phi(x)$ is continuous over (1). Moreover, we readily show that

$$\lim_{h \to 0} \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2} = 2k.$$

Hence, at any point x of (1) this expression is positive if h is sufficiently small. Now $\phi(x)$ is continuous on (1) and hence * has a greatest value on (1). Suppose this at x_1 and suppose $\phi(x_1) > 0$. Then x_1 is neither α nor β, and for sufficiently small values of h

$$\phi(x_1 + h) - 2\phi(x_1) + \phi(x_1 - h) \leqq 0.$$

This is contrary to what was shown above. Hence $\phi(x_1) \leqq 0$. In other words, $\phi(x) \leqq 0$ at all points of (1). This is true no matter which sign is taken preceding the expression defining $\phi(x)$. But $k(x-\alpha)(x-\beta)$ can be made numerically as small as desired by taking k small enough. Consequently

$$F(x) - F(\alpha) - \frac{x-\alpha}{\beta-\alpha}[F(\beta) - F(\alpha)] \equiv 0,$$

which establishes the theorem.

Theorem 202. HYPOTHESIS:

$$(2) \qquad \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

converges to zero at all points. CONCLUSION:

$$\alpha_0 = \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \ldots = 0.$$

PROOF: Let

$$F(x) = \frac{\alpha_0 x^2}{4} - \frac{\alpha_1}{1^2} \cos x - \frac{\beta_1}{1^2} \sin x - \frac{\alpha_2}{2^2} \cos 2x - \frac{\beta_2}{2^2} \sin 2x - \ldots.$$

Consider

$$(3) \qquad \frac{F(x+h) - 2F(x) + F(x-h)}{h^2}.$$

We can show that this approaches zero when h approaches zero. To do this, substitute, use the elementary addition formulas and show that (3), as a series in h, converges uniformly when $|h| < r < 1$ and then let $h \to 0$. It results that $F(x)$ is linear; that is $\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ are all zero.

* See, for example, Hobson, *Theory of Functions of a Real Variable*, p. 224, ed. 1907.

Corollary. *If (2) converges at all points to the same values as does the Fourier series corresponding to $f(x)$, then (2) is the Fourier series corresponding to $f(x)$.*

§ 6. The Fourier constants.

Theorem 203. HYPOTHESIS: *$f(x)$ is defined at every point of $-\pi \leqq x \leqq \pi$ and is of bounded variation on that interval.*

CONCLUSION:

$$|a_n| = \left|\int_{-\pi}^{\pi} f(x) \cos nx \, dx\right| < \frac{M}{n}$$

and

$$|b_n| = \left|\int_{-\pi}^{\pi} f(x) \sin nx \, dx\right| < \frac{M}{n},$$

where M is a constant.

PROOF: We can write $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ and $f_2(x)$ are defined at all points of the interval and are monotonic. By the second mean value theorem,

$$\int_{-\pi}^{\pi} f_1(x) \cos nx \, dx$$

$$= f_1(-\pi) \int_{-\pi}^{\xi} \cos nx \, dx + f_1(\pi) \int_{\xi}^{\pi} \cos nx \, dx$$

$$= [f_1(-\pi) - f_1(\pi)] \frac{\sin n\xi}{n}.$$

Hence,

$$\left|\int_{-\pi}^{\pi} f_1(x) \cos nx \, dx\right| < \frac{M_1}{n}$$

and similarly

$$\left|\int_{-\pi}^{\pi} f_2(x) \cos nx \, dx\right| < \frac{M_2}{n}.$$

Let $M = M_1 + M_2$ and $|a_n| < \dfrac{M}{n}$. In like manner,

$$|b_n| < \frac{M}{n}.$$

Corollary 1. $a_n \to 0$, $b_n \to 0$.

Corollary 2. *If $p > 1$, $\sum_{n=1}^{\infty} a_n^p$ and $\sum_{n=1}^{\infty} b_n^p$ both converge.*

EXERCISES

307–312. Obtain Fourier series which converge to each of the following functions at their points of continuity on the interval $-\pi < x < \pi$:

$$x, \quad x^2, \quad xe^x, \quad x^2e^x, \quad -\tfrac{1}{4}x^2 + \tfrac{1}{12}\pi^2, \quad \tfrac{1}{12}(\pi^2 x - x^3),$$

$$f(x) = \begin{cases} 1, & -\pi \leqq x \leqq 0 \\ -1, & 0 < x \leqq \pi \end{cases}, \quad f(x) = \begin{cases} x, & -\pi \leqq x \leqq 0 \\ -x, & 0 < x \leqq \pi \end{cases}.$$

313–320. Establish the following relations, determining in each instance the points at which the use of the equality mark is justified.

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \tfrac{1}{2}(\pi - x),$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \tfrac{1}{4}(\pi - x)^2 - \tfrac{1}{12}\pi^2,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(2n-1)x}{(2n-1)^4} = \frac{\pi x}{8}\left(\frac{\pi^2}{4} - \frac{x^3}{3}\right),$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2n-1)x}{2n-1} = \frac{\pi}{4},$$

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} = \frac{\pi x}{\text{[illegible]}}(\pi - x),$$

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2} = (x^2 - x + \tfrac{1}{6})\pi^2,$$

$$\sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n^3} = (\tfrac{2}{3}x^3 - x^2 + \tfrac{1}{3}x)\pi^3,$$

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^4} = (-\tfrac{1}{3}x^4 - \tfrac{2}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{1}{90})\pi^4.$$

321. Obtain Fourier series different from those obtained in Exercises 307–312 but which nevertheless represent the functions of those exercises over part of the interval, $-\pi < x < \pi$. State the intervals in which the series represents the function in each case.

CHAPTER XVII

SUMMATION OF DIVERGENT SERIES

§ 1. General Discussion.

In Chapter II an infinite series was defined as an unending succession of terms as $a_0, a_1, a_2, \ldots$, and then it was pointed out that we write it with plus signs

$$a_0 + a_1 + a_2 + \ldots,$$

or the equivalent $\sum_{n=0}^{\infty} a_n$ in deference to custom.

Moreover, the series has been called convergent in case a certain limit, namely $\lim_{n\to\infty} s_n$, exists, and we associate this limit with the series by calling it its sum. Now there is no *a priori* reason why some other limit might not be associated with the series; and if we find one which arises naturally in mathematical operations, why this new limit may not be useful, be studied by means of the series, or in turn bring new light to bear on the series itself. This is a particularly interesting problem in case the series is divergent, for here the sum, $\lim_{n\to\infty} s_n$, is non-existent. Moreover, there are ways to get $s = \lim_{n\to\infty} s_n$ for a convergent series apart from studying $\lim_{n\to\infty} s_n$ directly. For example, we learned in Theorem 80 that

$$\lim_{n\to\infty} \frac{s_0 + \ldots + s_n}{n+1} = s.$$

Now an interesting thing is that for certain divergent series

$$\lim_{n\to\infty} \frac{s_0 + \ldots + s_n}{n+1}$$

exists. A classical and perhaps the simplest example is the series $1 - 1 + 1 - 1 + 1 - \ldots$. Here

$$\lim_{n\to\infty} \frac{s_0 + \ldots + s_n}{n+1} = \frac{1}{2}$$

and this seems a natural number to associate with the series, for: (i) the usual process of long division applied to $\frac{1}{1+x}$ formally yields $1 - x + x^2 - x^3 + \ldots$, as does development by Maclauren's formula; (ii)

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

when $|x| < 1$, the series being convergent and the equality being used in the usual sense; (iii) when $0 < x < 1$

$$\lim_{x\to 1} (1 - x + x^2 - x^3 + \ldots) = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}.$$

Definition 41. *Given the series*

(1) $$a_0, a_1, \ldots,$$

where $s_n = a_0 + \ldots + a_{n-1}$, *if* $\lim_{n\to\infty} \frac{s_0 + \ldots + s_n}{n+1}$ *exists and equals* S, *(1) is said to be summable to* S *by the method of the Arithmetic Mean.*

Now there certainly seems no reason why still other methods may not be devised for associating a limit (sum) with a divergent series. Many such have been suggested and studied. We shall give a few, and make some comment and study of some of them.

§ 2. The Hölder and Cesàro Definitions.

The Hölder definition is:

Definition 42. *Let* $H_n^{(0)} = s_n$ *defined in the ordinary way,*

$$H_n^{(1)} = \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(0)},$$

$$H_n^{(2)} = \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(1)},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(1) $$H_n^{(r)} = \frac{1}{n+1} \sum_{n_1=0}^{n} H_{n_1}^{(r-1)} = \frac{1}{n+1} \sum_{n_1=0}^{n} \cdots \frac{1}{n_{r-1}+1} \sum_{n_r=0}^{n_{r-1}} s_{n_r}.$$

Then $\lim_{n\to\infty} H_n^{(r)} = S \equiv sum.$

The Cesàro definition is:

Definition 43. *Let* $C_n^{(r)} = \dfrac{S_n^{(r)}}{D_n^{(r)}}$, *r being an integer* $\geqq 0$, *and*

$$S_n^{(r)} = s_n + rs_{n-1} + \frac{r(r+1)}{2!} s_{n-2} + \dots + \frac{r(r+1)\dots(r+n-1)}{n!} s_0$$

and
$$D_n^{(r)} = \frac{(r+1)(r+2)\dots(r+n)}{n!}.$$

Then, $\lim_{n\to\infty} C_n^{(r)} = S \equiv$ *sum.*

We shall see * that this definition gives the same value to $C_n^{(r)}$ as the following perhaps more natural one,

$$(2) \qquad C_n^{(r)} = \frac{r!}{(n+1)\dots(n+r)} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \dots \sum_{n_r=0}^{n_{r-1}} s_{n_r}.$$

Definition 44. *In case* $C_n^{(r)}$ *approaches a limit S, the series is said to be summable (Cesàro) of order r to S. In case* $H_n^{(r)} \to S$ *the series is said to be summable* (Hölder) *of order r to S.*

Theorem 204.

$$D_n^{(r)} = 1 + r + \frac{r(r+1)}{2!} + \dots + \frac{r(r+1)\dots(r+n-1)}{n!}$$

and
$$S_n^{(r+1)} = S_0^{(r)} + S_1^{(r)} + \dots + S_n^{(r)}.$$

Proof is elementary by mathematical induction or in any of a variety of ways and is not given here. [See § 6.]

The denominator in $C_n^{(r)}$ then appears as the sum of the coefficients in the numerator.

We observe that both the Cesàro and Hölder definitions are generalizations of the arithmetic mean. The similarity of formulas (1) and (2) is obvious. The resemblance can immediately be carried further. We see that

$$C_n^{(r)} = u_r^{(0)}(n)\, s_0 + \dots + u_r^{(n)}(n)\, s_n$$

and that $H_n^{(r)}$ equals an expression in the same form. We

* See formula (7) below from which this follows.

observe, moreover, that either $H_n^{(r)}$ or $C_n^{(r)}$ can be written in the form $v_r^{(0)}(n)\, a_0 + \dots + v_r^{(n)}(n)\, a_n$. We shall pursue the subject along the lines of these ways of writing the partial sum later (§ 13). For the present we shall make a somewhat detailed study of these two methods of summation from the definitions first given.

Definition 45. *A method of summation will be called regular if it sums a convergent series to the usual sum. That is, if S is the sum obtained for a convergent series by the method in question,* $S = \lim_{n\to\infty} s_n$, *where* s_n *has the usual significance.*

There are many places where this property of a method of summation is valuable, particularly in matters of analytic continuation, where it is desired to extend the domain of a function defined by a convergent series beyond the region of convergence. Regularity is usually imposed as a necessary condition on a method of summation before the method is considered worthy of study.

Theorem 205. *The Hölder method of summation is regular.*

PROOF: Refer to Theorem 80 and repeat the proof given there r times, that is, use mathematical induction.

A proof of the regularity of the Cesàro definition from formula (2) is simple but is not given inasmuch as it will appear that the regularity follows from that of the Hölder definition.

Definition 46. *Two methods of summation will be said to be equivalent if every series summable by the one method is also summable by the other and to the same value.*

Theorem 206. *The Hölder and Cesàro methods of summation are equivalent.*

PROOF: Let $H_n^{(r)}$ and $C_n^{(r)}$ have the same meaning as previously.* We readily show that

* Definitions 42 and 43.

$$(3)\quad C_n^{(r)} = s_0 \frac{r}{r+n} + \dots + s_k \frac{r(n-k+1)\dots n}{(r+n-k)\dots(r+n)} + \dots + s_n \frac{r\,.\,(n!)}{r\dots(r+n)}$$

and then verify that $C_n^{(r)}$ satisfies

$$(4)\qquad (n+r+1)\,C_n^{(r+1)} - n\,C_{n-1}^{(r+1)} = (r+1)\,C_n^{(r)},$$

which may be written

$$(5)\qquad \Delta\,\{n\,C_{n-1}^{(r+1)}\} + r\,C_n^{(r+1)} = (r+1)\,C_n^{(r)} \quad \text{or}$$

$$(6)\qquad (n+1)\,C_n^{(r+1)} + r\sum_{n=0}^{n} C_n^{(r+1)} = (r+1)\sum_{n=0}^{n} C_n^{(r)}.$$

By solving (4) we get the following.

$$(7)\quad C_n^{(r+1)} = \frac{1\,.\,2\dots n}{(r+2)\dots(r+n+1)}\sum_{n=0}^{n}\frac{(r+1)\dots(r+n)}{n!}\,C_n^{(r)}$$

$$= \frac{(r+1)!}{(n+1)\dots(n+r+1)}\sum_{n=0}^{n}\frac{(n+1)\dots(n+r)}{1\,.\,2\dots r}\,C_n^{(r)}.$$

We shall now establish by mathematical induction the fundamental formula,

$$(8)\quad H_n^{(r)} = k_0^{(r)}\,C_n^{(r)} + k_1^{(r)}\,.\,\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)} + k_2^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)} + \dots + k_{r-1}^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\dots\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)},$$

where $\quad k_0^{(r)} + \dots + k_{r-1}^{(r)} = 1.$

Assume (8), then

$$(9)\quad H_n^{(r+1)} = \frac{1}{n+1}\sum_{n=0}^{n} H_n^{(r)} = k_0^{(r)}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)} + k_1^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)}$$

$$+ k_2^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)} + \dots + k_{r-1}^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\dots\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)}.$$

Substitute for $\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)}$ in each sum of (9) its value from (6). Take for example

$$(10)\quad k_1^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r)}$$

$$= k_1^{(r)}\frac{1}{n+1}\sum_{n=0}^{n}\left[\frac{1}{r+1}C_n^{(r+1)} + \frac{r}{r+1}\cdot\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r+1)}\right]$$

$$= k_1^{(r)}\frac{1}{r+1}\cdot\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r+1)} + k_1^{(r)}\cdot\frac{r}{r+1}\cdot\frac{1}{n+1}\sum_{n=0}^{n}\frac{1}{n+1}\sum_{n=0}^{n} C_n^{(r+1)}.$$

We notice that $\frac{k_1^{(r)}\,1}{r+1} + \frac{k_1^{(r)}\,r}{r+1} = k_1^{(r)}$. We then notice that we have an expression of the same form as (8) but with r replaced by $r+1$. As $H_n^{(1)} = C_n^{(1)}$, proof of the formula follows by induction.

We next prove by mathematical induction the general formula

$$(11)\quad C_n^{(r)} = h_0^{(r)}H_n^{(r)} + h_1^{(r)}\frac{1\,.\,2\dots r}{(n+1)\dots(n+r)}\sum_{n=0}^{n}\frac{(n+1)\dots(n+r-1)}{(r-1)!}H_n^{(r)}$$

$$+ h_2^{(r)}\frac{1\,.\,2\dots r}{(n+1)\dots(n+r)}\sum_{n=0}^{n}\sum_{n=0}^{n}\frac{(n+1)\dots(n+r-2)}{(r-2)!}H_n^{(r)} + \dots + h_{r-1}^{(r)}\frac{1\,.\,2\dots r}{(n+1)\dots(n+r)}\sum_{n=0}^{n}\dots\sum_{n=0}^{n}(n+1)\,H_n^{(r)},$$

where $h_0^{(r)} + h_1^{(r)} + \dots + h_{r-1}^{(r)} = 1.$

To prove this formula, substitute in (7) and then make the substitution

$$\sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!} H_n^{(r)} \tag{12}$$

$$= \frac{(n+1)\dots(n+r-1)}{(r-2)!} \cdot \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$

$$- \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-3)!} \cdot \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$

$$= (r-1) \frac{(n+1)\dots(n+r-1)}{(r-1)!} H_n^{(r+1)}$$

$$-(r-2) \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!} H_n^{(r+1)},$$

and similarly for other sums. To prove (12), &c., sum by parts using the formula

$$\sum_{n=0}^{n} u(n)\, \Delta v(n) = u(n)\, v(n)\Big]_0^{n+1} - \sum_{n=0}^{n} v(n+1)\, \Delta u(n).$$

Notice that the coefficients add to unity and then by mathematical induction formula (11) is proved.

The conclusions of the theorem are readily drawn from equations (8) and (11). Consider first that $C_n^{(r)} \to s$. Then each sum in (8) approaches s; and hence,

$$H_n^{(r)} \to (k_0^{(r)} + \dots + k_{r-1}^{(r)})\, s = s.$$

The proof that each sum approaches s is simply a repetition of the arithmetic mean theorem number 80. Next suppose that $H_n^{(r)} \to s$. Let $H_n^{(r)} = s + \delta$, where $\delta \to 0$. Suppose $|\delta| < G$ always and let $\epsilon > 0$ be a given constant. Suppose that when $n > N$, $|\delta| < \epsilon$. Then any sum as

$$\frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=0}^{n} \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!}\, s = s$$

by actual summation; also

$$\left| \frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=0}^{N} \sum_{0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!}\, \delta \right|$$

$$\leq G \frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=0}^{N} \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!} \to 0.$$

$$\left| \frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=N}^{n} \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!}\, \delta \right|$$

$$\leq \frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=0}^{n} \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!}\, \epsilon = \epsilon.$$

It follows that

$$\frac{1 \cdot 2 \dots r}{(n+1)\dots(n+r)} \sum_{n=0}^{n} \sum_{n=0}^{n} \frac{(n+1)\dots(n+r-2)}{(r-2)!} H_n^{(r)} \to s.$$

This is a representative term and proof of the theorem is complete.

The exact values of the coefficients $k_0^{(r)}, k_1^{(r)}, \dots, k_{r-1}^{(r)}$ were not necessary to the proof of the theorem. However, the following difference equation which results from the manner of their formation (see equation (10)) allows us to solve for as many of them as desired.

$$k_i^{(r+1)} = \frac{1}{r+1} k_i^{(r)} + \frac{r}{r+1} k_{i-1}^{(r)}, \quad r > 0. \tag{13}$$

We notice for example that

$$k_{-1}^{(r)} = 0, \quad k_0^{(0)} = 1, \quad k_0^{(r)} = \frac{1}{r!}.$$

Rewrite (13) as

$$\Delta_r \{r!\, k_i^{(r)}\} = r\,(r!)\, k_{i-1}^{(r)}$$

or letting $r!\, k_i^{(r)} = j_i^{(r)}$ we have

$$j_i^{(r)} = \sum_{r=0}^{r-1} r j_{i-1}^{(r)}.$$

Similarly the coefficients h satisfy the following difference equation:

$$h_i^{(r+1)} = (r-1)\, h_i^{(r)} - (r-2)\, h_{i-1}^{(r)}.$$

§ 3. Summation by the Hölder-Cesàro method.

Theorem 207. HYPOTHESIS: $H_n^{(r)} \to s$. CONCLUSION:

$$H_n^{(r+1)} \to s.$$

PROOF: This theorem follows from Definition 42 and Theorem 80.

Theorem 208. HYPOTHESIS: $C_n^{(r)} \to s$. CONCLUSION: $C_n^{(r+1)} \to s$.

Proof follows from Theorems 206 and 207. An independent proof could readily be given, however, by means of formula (7) of section 2.

Definition 47. *A series is said to be r-fold indeterminate if $C_n^{(r)}$ formed for it approaches a limit but $C_n^{(r-1)}$ does not.*

It is frequently said to be summable $C^{(r)}$ in case $C_n^{(r)}$ approaches a limit.

Theorem 209. *A necessary condition, for any particular series, that $C_n^{(r)} \to$ a limit, is that* $\frac{1}{n} C_n^{(r-1)} \to 0$.

PROOF: From equation (4) of section 2,

$$C_n^{(r)} - \frac{n}{n+r} C_{n-1}^{(r)} = \left(\frac{rn}{n+r}\right) \frac{1}{n} C_n^{(r-1)}.$$

Supposing that $C_n^{(r)}$ approaches a limit, the left-hand member approaches zero, and $\frac{rn}{n+r}$ approaches r. The theorem follows.

Theorem 210. *A necessary condition that $\sum_{n=0}^{\infty} a_n$ be summable $C^{(r)}$ is that* $\frac{s_n}{n^r} \to 0$.

PROOF: We write

$$\frac{1}{n} C_n^{r-1} - \frac{n}{n+r-1} \cdot \frac{1}{n} C_n^{(r-1)} = \frac{(r-1)\,n}{n+r-1} \cdot \frac{1}{n^2} C_n^{(r-2)}.$$

The left-hand member approaches zero, hence $\frac{1}{n^2} C_n^{(r-2)}$ does also. We continue in like manner writing the difference equation between $C_n^{(r-2)}$ and $C_n^{(r-3)}$, &c., arriving at the result of the theorem.

Corollary. *A necessary condition that $\sum_{n=0}^{\infty} a_n$ be summable $C^{(r)}$ is that* $\frac{a_n}{n^r} \to 0$.

PROOF: $$\frac{a_n}{n_r} = \frac{s_n}{n^r} - \frac{s_{n-1}}{n^r} \to 0.$$

The last two theorems could be even more simply proved for $H_n^{(r)}$ by means of the relation

$$(n+1)\, H_n^{(r+1)} - n H_{n-1}^{(r+1)} = H_n^{(r)}.$$

Theorem 211. HYPOTHESIS: *$\sum_{n=0}^{\infty} a_n$ is summable $C^{(r)}$ to s.*

CONCLUSION: *$\sum_{n=1}^{\infty} a_n$ is summable $C^{(r)}$ to $s - a_0$.*

Proof is easy, particularly from the Hölder formula and is omitted.

§ 4. Multiplication of series.

Theorem 212. HYPOTHESES: (i) *$\sum_{n=0}^{\infty} a_n$ is Cesàro summable of order k to A;* (ii) *$\sum_{n=0}^{\infty} b_n$ of order l to B;* (iii)

$$c_n = a_n b_0 + a_{n-1} b_1 + \dots + a_0 b_n = \sum_{a=0}^{n} a_a b_{n-a}.$$

CONCLUSION: *$\sum_{n=0}^{\infty} c_n$ is summable of order $k+l+1$ to AB.*

PROOF: Let $A_n^{(k-1)} = \sum_{n_1=0}^{n} \dots \sum_{n_k=0}^{n_{k-1}} a_{n_k}$,

$$B_n^{(l-1)} = \sum_{n_1=0}^{n} \dots \sum_{n_l=0}^{n_{l-1}} b_{n_l} \text{ and } C_n^{(j-1)} = \sum_{n_1=0}^{n} \dots \sum_{n_j=0}^{n_{j-1}} c_{n_j}.$$

Then,

(1) $$C_n^{(k+l+1)} = A_n^{(k)} B_0^{(l)} + A_{n-1}^{(k)} B_1^{(l)} + \dots + A_0^{(k)} B_n^{(l)}.$$

To prove this we proceed in a manner similar to that under Theorem 82.

$$C_n^{(0)} = \sum_{n=0}^{n} c_n = \sum_{a=0}^{n} a_a (b_0 + \dots + b_{n-a}) = \sum_{a=0}^{n} a_a B_{n-a}^{(0)}.$$

This is the same form as c_n with b_{n-a} replaced by $B^{(1)}_{n-a}$. From symmetry we also can write

$$C_n^{(0)} = \sum_{a=0}^{n} b_a A^{(0)}_{n-a} \quad \text{or} \quad \sum_{a=0}^{n} a_{n-a} B^{(0)}_a.$$

We repeat this process, obtaining first

$$C_n^{(1)} = \sum_{a=0}^{n} A^{(0)}_a B^{(0)}_{n-a}$$

and finally arriving at (1).

We now assume that

$$\frac{k!}{(n+1)\ldots(n+k)} A_n^{(k)} \longrightarrow 0 \quad \text{and} \quad \frac{l!}{(n+1)\ldots(n+l)} B_n^{(l)} \longrightarrow 0.$$

[See (2) of § 2 of this chapter.] To accomplish this we replace a_0 by $\bar{a}_0 = a_0 - A$ and b_0 by $\bar{b}_0 = b_0 - B$. [See Theorem 211.] We now write (1) as follows and use a $(-)$ indicating the presence of $\bar{a}_0$ and $\bar{b}_0$.

$$(2) \quad \frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)} \bar{C}_n^{(k+l+1)} = \frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)}$$

$$\cdot \left[\frac{(n+1)\ldots(n+k)}{k!}\left(\frac{k!}{(n+1)\ldots(n+k)}\bar{A}_n^{(k)}\right)\cdot\frac{l!}{l!}(\bar{B}_0^{(l)})\right.$$

$$+\frac{n(n+1)\ldots(n+k-1)}{k!}\left(\frac{k!}{n(n+1)\ldots(n+k-1)}\bar{A}^{(k)}_{n-1}\right)$$

$$\cdot\frac{2\cdot 3\ldots(l+1)}{l!}\left(\frac{l!}{2\cdot 3\ldots(l+1)}\bar{B}_1^{(l)}\right)+\ldots$$

$$\left.+\frac{k!}{k!}(\bar{A}_0^{(k)})\cdot\frac{(n+1)\ldots(n+l)}{l!}\left(\frac{l!}{(n+1)\ldots(n+l)}\bar{B}_n^{(l)}\right)\right].$$

Take $n > M$ which is chosen so large that when $n > \frac{M}{2}$

$$\left|\frac{k!}{(n+1)\ldots(n+k)}\bar{A}_n^{(k)}\right| \quad \text{and} \quad \left|\frac{l!}{(n+1)\ldots(n+l)}\bar{B}_n^{(l)}\right|$$

are both less than $\delta > 0$, and suppose each of them less than g always. Then,

$$\left|\frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)}\bar{C}_n^{(k+l+1)}\right| \leqq \frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)}$$

$$\delta g \frac{1}{l!}\frac{1}{k!}\cdot[(n+1)\ldots(n+k)(l!)+n(n+1)\ldots(n+k-1)\cdot 2\cdot 3$$

$$\ldots(l+1)+\ldots+(k!)(n+1)\ldots(n+l)] < \frac{(k+l+1)!}{(l!)(k!)} g\delta$$

$$\cdot\frac{(n+1)[(n+1)\ldots(n+k)][(n+1)\ldots(n+l)]}{(n+1)\ldots(n+k+l+1)} < S\delta,$$

where S is a constant since

$$\frac{(n+1)[(n+1)\ldots(n+k)][(n+1)\ldots(n+l)]}{(n+1)\ldots(n+k+l+1)} \longrightarrow 1.$$

But $S\delta < \epsilon$ (preassigned) if $\delta < \frac{\epsilon}{S}$ hence $\bar{C}_n^{(k+l+1)} \longrightarrow 0$. Now replace $\bar{a}_0$ by $a_0 + A$ and $\bar{b}_0$ by $b_0 + B$ in (2).

$$\frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)} \bar{C}_n^{(k+l+1)}$$

is increased by AB for c_n is increased by AB to which if we apply the operator

$$\frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)} \sum_{n_1=1}^{n} \ldots \sum_{n_{k+l+1}=1}^{n_{k+l}}$$

we get AB. The operator is distributive. Hence

$$\frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)} C_n^{(k+l+1)}$$

$$= \frac{(k+l+1)!}{(n+1)\ldots(n+k+l+1)} \bar{C}_n^{(k+l+1)} + AB \longrightarrow AB.$$

This completes the proof.

§ 5. The boundary value condition.

We introduce at this point a lemma for which we shall find occasion for immediate as well as not infrequent subsequent use.

Theorem 213 (*Lemma*). HYPOTHESES: (i) $\lim_{n\to} s_n = s$; (ii) $a_n^{(p)} > 0$, p *real*; (iii) $\lim_{p\to\infty} \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} = 0$, *m being any fixed positive integer. The fraction is, of course, defined for all values of p in question.* CONCLUSION:

$$\lim_{p\to\infty} \frac{\sum_{n=0}^{\infty} a_n^{(p)} s_n}{\sum_{n=0}^{\infty} a_n^{(p)}} \equiv \lim_{p\to\infty} S_p = s.$$

PROOF: Let $D_p = S_p - s$. We shall show that $D_p \to 0$. Let $s_n = s + \epsilon_n$. Then,

$$D_p = \frac{\sum_{n=0}^{m} (s_n - s)\, a_n^{(p)} + \sum_{n=m+1}^{\infty} \epsilon_n a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}}.$$

Next indicate by g_m a positive number such that $g_m \geqq |s_i|$, $i = 0, 1, 2, \ldots, m$. Whereupon

$$|D_p| \leqq (g_m + |s|) \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} + \frac{\sum_{n=m+1}^{\infty} |\epsilon_n|\, a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}}.$$

Now, let an $\epsilon > 0$ be given and choose m so large that $|\epsilon_n| < \epsilon$ when $n > m$. We then have

$$\sum_{n=m+1}^{\infty} |\epsilon_n|\, a_n^{(p)} < \epsilon \left[\sum_{n=0}^{\infty} a_n^{(p)} - \sum_{n=0}^{m} a_n^{(p)} \right].$$

From which

$$|D_p| < (g_m + |s|) \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} + \epsilon \left[1 - \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} \right] \to \epsilon,$$

which is arbitrarily small.. Hence $D_p \to 0$.

Many divergent series as $\sum^{\infty} a_n$ which are summable by methods in common use are such that the corresponding power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

has a radius of convergence equal to 1.

Definition 47. *If for all such series where a particular method of summation gives the value s,* $\lim^*_{x\to 1-0} f(x) = s$ *the method of summation in question is said to obey the boundary value condition.*

Theorem 214. *The Cesàro method of summation obeys the boundary value condition.*

PROOF: Let s_n of the lemma be $\frac{S_n^{(r)}}{D_n^{(r)}}$ of section 2. Put

$$x = 1 - \frac{1}{p}$$

so that when x varies from a to **1** $(0 < a < x)$ p increases indefinitely from $\frac{1}{1-a}$. Also take

$$a_n^{(p)} = D_n^{(r)} \left(1 - \frac{1}{p}\right)^n$$

The expression S_p of the lemma then becomes

$$S_p = \frac{\sum_{n=0}^{\infty} S_n^{(r)} \left(1 - \frac{1}{p}\right)^n}{\sum_{n=0}^{\infty} D_n^{(r)} \left(1 - \frac{1}{p}\right)^n} = \frac{\sum_{n=0}^{\infty} S_n^{(r)} x^n}{\sum_{n=0}^{\infty} D_n^{(r)} x^n}$$

$$= \frac{(1-x)^{-(r+1)} \sum_{n=0}^{\infty} a_n x^n \,§}{(1-x)^{-(r+1)}} = \sum_{n=0}^{\infty} a_n x^n,$$

* 1 — 0 means that the approach is from below with x real.

§ See (6) and (4) of § 6 below.

so that
$$\lim_{p\to\infty} S_p = \lim_{x\to 1-0} \sum_{n=0}^{\infty} a_n x^n.$$

Confine p to integral values greater than one. Hypothesis (iii) of the lemma is now satisfied since

$$\lim_{p\to\infty}\left[\sum_{n=0}^{m} D_n^{(r)}\left(1-\frac{1}{p}\right)^n \Big/ \sum_{n=0}^{\infty} D_n^{(r)}\left(1-\frac{1}{p}\right)^n\right]$$

$$= \lim_{p\to\infty} \frac{1+\dfrac{(r+1)}{1}\left(1-\dfrac{1}{p}\right)+\dfrac{(r+1)(r+2)}{2\,!}\left(1-\dfrac{1}{p}\right)^2+\dots}{1+\dfrac{(r+1)}{1}\left(1-\dfrac{1}{p}\right)+\dfrac{(r+1)(r+2)}{2\,!}\left(1-\dfrac{1}{p}\right)^2+\dots},$$

where the numerator is to $(m+1)$ terms only. This limit is zero since the denominator has a meaning for all $p>1$ but becomes infinite with p while the numerator remains finite. Apply the lemma and the theorem is proved.

§ 6. Testing particular series.

In this section we shall treat briefly the question of testing individual series for summability by the Cesàro method. We develop a number of useful identities and prove two theorems.

By definition

(1) $$s = \lim_{n\to\infty} \frac{S_n^{(r)}}{D_n^{(r)}}, \text{ where}$$

(2) $$S_n^{(r)} = s_n + r s_{n-1} + \frac{r(r+1)}{2\,!} s_{n-2} + \dots + \frac{r(r+1)\dots(r+n-1)}{n\,!} s_0,$$

and

(3) $$D_n^{(r)} = \frac{(r+1)(r+2)\dots(r+n)}{n\,!}.$$

Now when $|x|<1$

(4) $$(1-x)^{-r} = 1 + rx + \frac{r(r+1)}{2\,!}x^2 + \dots.$$

Multiply by $\sum_{n=0}^{\infty} s_n x^n$ and we have

$$\sum_{n=0}^{\infty} S_n^{(r)} x^n = (1-x)^{-r} \sum_{n=0}^{\infty} s_n x^n.$$

Next,

(5) $$\sum_{n=0}^{\infty} s_n x^n = (1+x+x^2+\dots) \sum_{n=0}^{\infty} a_n x^n$$

by multiplication and hence

(6) $$\sum_{n=0}^{\infty} S_n^{(r)} x^n = (1-x)^{-(r+1)} \sum_{n=0}^{\infty} a_n x^n.$$

Increase r by 1 and we have

(7) $$\sum_{n=0}^{\infty} S_n^{(r+1)} x^n = (1-x)^{-(r+2)} \sum_{n=0}^{\infty} a_n x^n.$$

From (6) and (7)

(8) $$\sum_{n=0}^{\infty} S_n^{(r+1)} x^n = (1-x)^{-1} \sum_{n=0}^{\infty} S_n^{(r)} x^n = (1+x+x^2+\dots)\sum_{n=0}^{\infty} S_n^{(r)} x^n,$$

so long as $|x|<1$. Equating coefficients

(9) $$S_n^{(r+1)} = S_0^{(r)} + \dots + S_n^{(r)} \text{ and consequently}$$
$$D_n^{(r+1)} = D_0^{(r)} + \dots + D_n^{(r)}$$

since $D_n^{(r)} = S_n^{(r)}$ with s_n replaced by 1. Proofs for formulas (9) by mathematical induction had already been suggested.

From the definition of s

$$S_n^{(r)} = sD_n^{(r)} + \epsilon_n D_n^{(r)} \text{ where } \epsilon_n \to 0.$$

We then write $S_n^{(r)} = sD_n^{(r)} + \rho_n$ and

(10) $$\sum_{n=0}^{\infty} S_n^{(r)} x^n = s\sum_{n=0}^{\infty} D_n^{(r)} x^n + \rho(x).$$

From which by (6) and (4), when $|x|<1$

(11) $$(1-x)^{-(r+1)} \sum_{n=0}^{\infty} a_n x^n = s(1-x)^{-(r+1)} + \rho(x).$$

Theorem 215. HYPOTHESIS: (11) *holds, where*

$$\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n \quad and \quad \frac{\rho_n}{D_n^{(r)}} \to 0.$$

CONCLUSION: *The series* $\sum_{n=0}^{\infty} a_n$ *is at most r-fold indeterminate.*

PROOF: Retrace steps. Go from (11) to (10). Then equate coefficients and get $S_n^{(r)} = sD_n^{(r)} + \rho_n$, where by hypothesis $\frac{\rho_n}{D_n^{(r)}} \to 0$. From this $\frac{S_n^{(r)}}{D_n^{(r)}} \to s$.

Theorem 216. *Relation* (11) *includes*

$$(12)\quad (1-x)^{-(r+1)} \sum_{n=0}^{\infty} a_n x^n = s\,(1-x)^{-(r+1)} + P_1(x)\,(1-x)^{-r}$$

and

$$(13)\quad (1-x)^{-(r+1)} \sum_{n=0}^{\infty} a_n x^n = s\,(1-x)^{-(r+1)} + P_2(x)\,(1-x^2)^{-r}$$

where P_1 *and* P_2 *are polynomials.*

PROOF: Suppose that there are p terms in $P_1(x)$ and that M is the largest of the absolute values of the coefficients. The coefficient of x^n in $P_1(x)\,(1-x)^{-r}$

$$= P_1(x)\left[1 + rx + \frac{r(r+1)}{2!}x^2 + \dots + \frac{r(r+1)\dots(r+n-1)}{n!}x^n + \dots\right]$$

is in absolute value less than $MpD_n^{(r-1)}$ and the quotient of this by $D_n^{(r)}$ is $\frac{Mpr}{r+n}$ which approaches zero as $n \to \infty$.

Similar reasoning applies to (13).

Formulas (12) and (13) are frequently useful in testing series for summability.

The following examples will clarify the whole situation:—

Ex. 1. $1-1+1-1+\dots$ is simply indeterminate.

Ex. 2. $1-2+3-4+\dots$ is two-fold indeterminate.

Ex. 3. $1-2^2+3^2-4^2+\dots$ is at most three-fold indeterminate. To prove this we write

$$1-2^2x+3^2x^2-4^2x^2+\dots = (1-x)\,(1+x)^{-3}.$$

Hence $(1-x)^{-4}(1-2^2x+3^2x^2-4^2x^3+\dots) = (1-x^2)^{-3}$, which satisfies (13) giving us $s = 0$.

Ex. 4. $1 - r + \frac{r(r+1)}{2!} - \frac{r(r+1)(r+2)}{3!} + \dots$

is at most r-fold indeterminate giving $s = \frac{1}{2^r}$, as is readily proved by (13) if we remember that for this series

$$\sum_{n=0}^{\infty} a_n x^n = (1+x)^{-r}.$$

Ex. 5. The theorem on multiplication (Theorem 212) may be useful thus

$$(1-1+1-1+\dots)(1-2+3-4+\dots) = 1-3+6-10+\dots$$

is surely at most four-fold indeterminate. It has the sum $\frac{1}{8}$.

§ 7. Uniform summability.

Definition 48. $\sum_{n=1}^{\infty} u_n(x)$ *where* $u_n(x)$ *is defined over a set I, is said to be uniformly summable over I to S in case* $C_n^{(r)} \to S$ *uniformly over I.*

A pursuit of this idea would simply carry us to a study of uniformly convergent sequences. The sequence could, if we like, be written as a series and the proofs of Chapter IX be applied. Such a treatment would involve repetition and will not be made.

§ 8. Fourier series.

One of the most interesting applications of the Cesàro (Hölder) method of summation is in the theory of Fourier series as is witnessed by Theorem 217. This theorem forms a natural completion to Theorem 196. Before proving it, however, we prove two lemmas.

Lemma 1. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 nt}{\sin^2 t}\,dt = n\frac{\pi}{2}.$

PROOF. $\dfrac{\sin^2 nt}{\sin t} = \sin t + \sin 3t + \ldots + \sin(2n-1)t$

when $t \neq k\pi$ and

$$\lim_{t \to k\pi} \frac{\sin^2 nt}{\sin t} = \lim_{t \to k\pi} (\sin t + \sin 3t + \ldots + \sin(2n-1)t) = 0.$$

But $\dfrac{\sin(2V-1)t}{\sin t} = 1 + 2\cos 2t + \ldots + 2\cos 2(V-1)t$

when $t \neq k\pi$ and the limits are equal when $t \to k\pi$. We substitute, integrate, and get the desired result.

Lemma 2. HYPOTHESIS: $\phi(t)$ is integrable from 0 to $\dfrac{\pi}{2}$ and $\lim\limits_{t \to 0} \phi(t) = 0$. CONCLUSION: $\dfrac{2}{n\pi}\displaystyle\int_0^{\frac{\pi}{2}} \phi(t) \frac{\sin^2 nt}{\sin^2 t} dt \to 0.$

PROOF. Let an $\epsilon > 0$ be given and choose $\delta < \dfrac{\pi}{2}$, such that $|\phi(t)| < \dfrac{\epsilon}{2}$ when $0 < t < \delta$. Then, using lemma 1,

$$(1) \qquad \left| \frac{2}{n\pi}\int_0^{\delta} \phi(t) \frac{\sin^2 nt}{\sin^2 t} dt \right| \leqq \frac{\epsilon}{2} \cdot \frac{2}{n\pi}\int_0^{\delta} \frac{\sin^2 nt}{\sin^2 t} dt < \frac{\epsilon}{2}.$$

Moreover if $M \geqq \phi(t)$

$$(2) \qquad \left| \frac{2}{n\pi}\int_{\delta}^{\frac{\pi}{2}} \phi(t) \frac{\sin^2 nt}{\sin^2 t} dt \right| \leqq \frac{2M}{n\pi} \cdot \frac{\pi}{2} \cdot \frac{1}{\sin^2 \delta} < \frac{\epsilon}{2}$$

when $n > \dfrac{2M}{\epsilon \sin^2 \delta}$. The lemma is proved by combining (1) and (2).

Theorem 217. HYPOTHESES: (i) *$f(x)$ is integrable $-\pi$ to π*; (ii)

$$\lim_{t \to 0} \tfrac{1}{2}[f(x_0 + 2t) + f(x_0 - 2t)] = s.$$

CONCLUSION: *The Fourier series for $f(x)$ is summable, C_1, at x_0 to s.*

PROOF: We already know

$$s_n(x_0) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}[f(x_0 + 2t) + f(x_0 - 2t)] \frac{\sin(2n+1)t}{\sin t} dt.$$

Consequently $\sigma_{n-1} = s_0 + s_1 + \ldots + s_{n-1}$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}[f(x_0 + 2t) + f(x_0 - 2t)] \frac{\sin^2 nt}{\sin^2 t} dt.$$

By lemma 1,

$$s = \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} s \frac{\sin^2 nt}{\sin^2 t} dt.$$

Hence,

$$\sigma_{n-1} - s = \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \left[\frac{f(x_0 + 2t) + f(x_0 - 2t)}{2} - s\right] \frac{\sin^2 nt}{\sin^2 t} dt \to 0$$

by lemma 2, completing the proof.

§ 9. Borel's first definition.

The Cesàro and Hölder methods of summation can be described as methods of the arithmetic mean, since they are both natural generalizations of the simple arithmetic mean method. In this section we shall give a very brief discussion of a method of summation due to Borel.*

Definition 49. *We define as sum of the series*

$$(1) \qquad \sum_{n=0}^{\infty} a_n$$

$$s = \lim_{\alpha \to \infty} e^{-\alpha} s(\alpha)$$

where $\qquad s(\alpha) = \displaystyle\sum_{n=0}^{\infty} \frac{s_n}{n!} \alpha^n$ and α is real.

We assume that this series is convergent for all values of α. The definition would, of course, be meaningless for a series for which this is not true.

This definition seems less natural than those of the arithmetic mean although if displayed in the following form it will appear somewhat similar to the Cesàro method

$$s = \lim_{\alpha \to \infty} \frac{s_0 + s_1 \dfrac{\alpha}{1} + s_2 \dfrac{\alpha^2}{2!} + s_3 \dfrac{\alpha^3}{3!} + \ldots}{1 + \dfrac{\alpha}{1} + \dfrac{\alpha^2}{2!} + \dfrac{\alpha^3}{3!} + \ldots}.$$

We prove the following theorem.

* *Leçons sur les séries divergentes,* chap. iii.

Theorem 218. *The method given by definition* 49 *is regular.* That is, in case of a convergent series it gives the usual sum.

To prove this theorem we have recourse to Theorem 213 (lemma). Let $p = \alpha \geqq 0$ range over the real numbers and let

$$a_n^{(p)} = \frac{\alpha^n}{n!}.$$

Then $$S_p = \frac{s_0 + s_1\alpha + s_2\frac{\alpha^2}{2!} + \dots}{1 + \alpha + \frac{\alpha^2}{2!} + \dots} = e^{-\alpha}s(\alpha).$$

Moreover $$\lim_{\alpha\to\infty} e^{-\alpha}\sum_{n=0}^{m}\frac{\alpha^n}{n!} = 0.$$

Hence the lemma yields the desired result.

This definition also satisfies the boundary value condition but proof will not be given here.

§ 10. Borel's integral definition.*

Definition 50. *Given an infinite series*

(1) $$\sum_{n=0}^{\infty} a_n.$$

Let $a(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \dots$, *where x is real.*

Then, $\int_0^\infty e^{-x}a(x)\,dx$ *will be called the sum of the series, provided that the integral exists.*

We shall make a somewhat more extended study of this method than of the last. The term summable will apply in this section to this definition only and the notation $\underset{n=1}{\overset{\infty}{\mathrm{S}}} a_n$ will at times be used in a manner quite parallel to the way in which we have been using $\sum_{n=1}^{\infty} a_n$ when matters of convergence were involved.

* *Leçons sur les séries divergentes*, p. 98.

Theorem 219. *The method of Definition* 50 *is regular.* (See Definition 44.)

PROOF: We assume (1) convergent with the sum s. Note first that $a_n \to 0$. Hence by Theorem 218 of the last section we have

(2) $$\lim_{p\to\infty} e^{-p}a(p) = \lim_{n\to\infty} a_n = 0.$$

Using the notation of that section again, by means of

$$\lim_{p\to\infty} e^{-p}s(p) = s$$

together with $[e^{-p}s(p)]_{p=0} = a_0$, we write

$$s - a_0 = \int_0^\infty \frac{d}{d\alpha}[e^{-\alpha}s(\alpha)]\,d\alpha.$$

But, $$\frac{d}{d\alpha}[e^{-\alpha}s(\alpha)] = e^{-\alpha}[s'(\alpha) - s(\alpha)],$$

where $$s'(\alpha) = s_1 + s_2\alpha + s_3\frac{\alpha^2}{2!} + \dots.$$

Moreover,

$$a'(\alpha) = s'(\alpha) - s(\alpha) = a_1 + a_2\alpha + a_3\frac{\alpha^2}{2!} + \dots.$$

From these, $$s - a_0 = \int_0^\infty e^{-\alpha}a'(\alpha)\,d\alpha.$$

Integrating by parts and making use of (2) we arrive at

$$\int_0^\infty e^{-\alpha}a(\alpha)\,d\alpha = s$$

as was desired.

Theorem 220. HYPOTHESIS: $\underset{n=0}{\overset{\infty}{\mathrm{S}}} a_n$ *and* $\underset{n=1}{\overset{\infty}{\mathrm{S}}} a_n$ *both exist and equal s and s' respectively.* CONCLUSION: $s = s' + a_0$.

PROOF: $s' = \int_0^\infty e^{-x}a'(x)\,dx$, where $a'(x) = \frac{d}{dx}a(x)$. We make the assumption that $a(x)$ is convergent for all values of x, as we have been doing. Term by term differentiation is justified by Theorem 143. Integrate the expression for s' by parts.

$$s' = \lim_{b\to\infty}[e^{-b}a(b)] - a_0 + s.$$

Hence [redacted] s and s' both exist $\lim\limits_{b\to\infty}[e^{-b}a(b)]$ exists. Moreover, $\lim\limits_{b\to\infty} e^{-b}a(b) = 0$ otherwise the integral s would not converge. Consequently, we have $s - s' = a_0$.

Theorem 221. HYPOTHESIS: *s' of the last theorem exists* CONCLUSION: [redacted] *also exists.*

PROOF: Let $y = \int_0^x e^{-\alpha}a(\alpha)\,d\alpha$, where x is a real variable. Then $\dfrac{dy}{dx} \equiv y' = e^{-x}a(x)$. Integrating by parts,

$$(3)\quad \int_{\text{[redacted]}}^x e^{-\alpha}a'(\alpha)\,d\alpha = \int_{\text{[redacted]}}^x e^{-\alpha}a(\alpha)\,d\alpha + y' - a_0$$
$$= y + y' - a_0 \longrightarrow s'.$$

We write $\eta = y - (a_0 + s')$ and shall prove $\eta \longrightarrow 0$. Let $\eta' = \dfrac{d\eta}{dx}$. Then $\eta + \eta' = y - (a_0 + s') + y' \longrightarrow 0$ by (3). Let

$$\eta' + \eta = f(x).$$

Then $f(x) \longrightarrow 0$. Let an [redacted] > 0 be given and choose M so that $|f(x)| < \epsilon$ when $x > M$.

By the formula for the solution of a linear differential equation of the first order,

$$\eta = e^{-x}\left[\int_1^x e^x f(x)\,dx + c\right].$$

From this $\quad |\eta| \leqq e^{-x}\int_1^x e^x|f(x)|\,dx + e^{-x}|c|$

$$\leqq e^{-x}\epsilon\int_1^x e^x dx + e^{-x}|c| \quad \text{when } x > M.$$

Hence, $\quad |\eta| \leqq \epsilon e^{-x}(e^x - e) + e^{-x}|c|$

$$\leqq \epsilon + Ke^{-x} < \delta$$

if $\epsilon < \dfrac{\delta}{2}$ and x is so large that $Ke^{-x} < \dfrac{\delta}{2}$. By definition

$$\eta = \int_{\text{[redacted]}}^x e^{-\alpha}a(\alpha)\,d\alpha - (a_0 + s').$$

Hence, $\quad \lim\limits_{x\to\infty}\int_0^x e^{-\alpha}a(\alpha)\,d\alpha = a_0 + s',$

completing the proof.

Corollary. *Any fixed number of terms may be prefixed to a summable series and the series will remain summable and the sums will be related as if the series were convergent.*

The converse of the last theorem and corollary is not true as is attested by the following example. Let $a(x) = e^x \sin e^x$. Develop by Maclauren's formula, drop the x's and we have the series to be summed. The exact form of this series does not concern us.

$$s = \int_0^\infty \sin e^x dx = \int_1^\infty \frac{\sin y}{y}\,dy$$
$$= \int_1^\pi \frac{\sin y}{y}\,dy + \int_\pi^{2\pi}\frac{\sin y}{y}\,dy + \int_{2\pi}^{3\pi}\frac{\sin y}{y}\,dy + \ldots,$$

which is a series with terms alternating in sign and after the first each less than the preceding and approaching zero. The series converges.

$$s' = \int_1^\infty \left(\frac{\sin y}{y} + \cos y\right)dy$$
$$= \int_1^\infty \frac{\sin y}{y}\,dy + \int_1^\infty \cos y\,dy.$$

The first integral converges. The second does not. Hence s' does not exist.

Two very easy theorems will now be stated without proof.

Theorem 222. $\quad \overset{\infty}{\underset{n=1}{\mathrm{S}}}(a_n \pm b_n) = \overset{\infty}{\underset{n=1}{\mathrm{S}}} a_n \pm \overset{\infty}{\underset{n=1}{\mathrm{S}}} b_n.$

Theorem 223. $\quad \overset{\infty}{\underset{n=1}{\mathrm{S}}} ca_n = c\,\overset{\text{[redacted]}}{\underset{n=1}{\mathrm{S}}} a_n.$

The following simple examples may serve to clarify the situation:—

Ex. 1. $\quad 1 - 1 + 1 - 1 + \ldots$

$$a(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = e^{-x},$$
$$\int_{\text{[redacted]}}^\infty e^{-x}a(x)\,dx = \int_0^\infty e^{-2x}dx = \tfrac{1}{2}.$$

Ex. 2. $1-t+t^2-t^3+\dots$

$$\int_0^\infty e^{-x}a(x)\,dx = \frac{1}{1+t}.$$

Ex. 3. $1-2+3-4+5-\dots$

$$a(x) = e^{-x}(1-x),$$

$$\int_0^\infty e^{-x}a(x)\,dx = \tfrac{1}{4}.$$

§ 11. Absolutely summable series.

The method of summation in this section is throughout the method of Definition 50 in the last section.

Definition 51. *A series*

(1) $$a_0+a_1+a_2+\dots$$

is called absolutely summable if the integrals

$$\int_0^\infty e^{-x}a(x)\,dx \quad and \quad \int_0^\infty e^{-x}\,|\,a^{(\lambda)}(x)\,|\,dx,$$

where $\lambda = 0,\ 1,\ \dots,$ represents an index of differentiation, all exist.

Theorem 224. HYPOTHESIS: *Series* (1) *is absolutely summable.* CONCLUSION: *$\mathop{\mathrm{S}}\limits_{n=\mu}^{\infty} a_n$ is also, $\mu \geqq 0$, and their sums are related as if the series were convergent.*

PROOF: The theorem is an immediate consequence of the fact that $a'(x)$ is $a(x)$ for the series formed by removing the first term of (1), and of Theorem 220.

Theorem 225. HYPOTHESIS: *$\mathop{\mathrm{S}}\limits_{n=1}^{\infty} a_n$ is absolutely summable.* CONCLUSION: *$\mathop{\mathrm{S}}\limits_{n=0}^{\infty} a_n$ is absolutely summable also.*

PROOF: We must prove that the existence of

$$\int_0^\infty e^{-x}\,|\,a'(x)\,|\,dx$$

implies the existence of

$$\int_0^\infty e^{-x}\,|\,a(x)\,|\,dx.$$

Let $\phi(x) = \int_0^x |\,a'(x)\,|\,dx \geqq \left|\int_0^x a'(x)\,dx\right| = |\,a(x)-a_0\,|.$

Hence $$|\,a(x)\,| \leqq \phi(x)+|\,a_0\,|.$$

Hence $$\int_0^\infty e^{-x}\,|\,a(x)\,|\,dx$$

converges if $$\int_0^\infty e^{-x}\phi(x)\,dx$$

does. We infer the convergence of this last from that of

$$\int_0^\infty e^{-x}\phi'(x)\,dx$$

by the arguments under Theorem 221. But the convergence of

$$\int_0^\infty e^{-x}\,|\,a'(x)\,|\,dx$$

implies the convergence of

$$\int_0^\infty e^{-x}\phi'(x)\,dx$$

since $\phi'(x) = |\,a'(x)\,|$. This together with 221 completes the proof.

An example of a series summable but not absolutely summable is where

$$a(x) = \frac{1}{x}[e^x(\cos x + i\sin x)-1].$$

$$\int_0^\infty e^{-x}a(x)\,dx \text{ exists.}$$

But $$|\,a(x)\,| \geqq \frac{e^x-1}{x}.$$

Hence $$\int_0^x e^{-x}\,|\,a(x)\,|\,dx \geqq \int_0^x \frac{1-e^{-x}}{x}\,dx \to \infty.$$

Theorem 226. HYPOTHESIS: *Series* (1) *is absolutely convergent.* CONCLUSION: *It is absolutely summable.*

PROOF: $\sum\limits_{n=0}^{\infty} |\,a_n\,|$ is convergent and hence

$$\int_0^\infty e^{-x}v^{(\lambda)}(x)\,dx = \sum_{n=\lambda+1}^{\infty} |\,a_n\,|$$

converges where $v(x)$ is built from $\sum_{n=0}^{\infty} |a_n|$ just as $a(x)$ is built from (1). But $|a^{(\lambda)}(x)| \leqq v^{(\lambda)}(x)$.

Hence
$$\int_0^{\infty} e^{-x} |a^{(\lambda)}(x)|\, dx \leqq \int_0^{\infty} e^{-x} v^{(\lambda)}(x)\, dx,$$
and consequently it converges.

§ 12. Uniform summability.

As in the last several sections we again confine our attention to Definition 50.

Definition 52. $\mathop{\mathrm{S}}\limits_{n=1}^{\infty} u_n(\alpha)$ *will be called uniformly summable for any set I of values of* α *if* $\int_0^{\infty} a_\alpha(x)\, e^{-x} dx$ *converges uniformly over I to a value* $U(x)$, that is, if given any $\epsilon > 0$, an X can be so chosen that when $x \geqq X$,
$$\left| \int_0^{x} e^{-x} a_\alpha(x)\, dx - U(\alpha) \right| < \epsilon$$
for every α of I simultaneously. $a_\alpha(x)$ bears the same familiar relation of Definition 50 to the series in question that $a(x)$ bears to (1).

Theorem 227. HYPOTHESIS: $u_n(\alpha)$ *is continuous at all points of a continuum I, and* $\mathop{\mathrm{S}}\limits_{n=1}^{\infty} u_n(\alpha)$ *is uniformly summable to* $U(\alpha)$ *over I.* CONCLUSION: $U(\alpha)$ *is continuous over I.*

PROOF: Consider any particular point α_0.
$$\left| \int_0^{\infty} e^{-x} a_\alpha(x)\, dx - \int_0^{\infty} e^{-x} a_{\alpha_0}(x)\, dx \right|$$
$$\leqq \left| \int_0^{X} e^{-x} a_\alpha(x)\, dx - \int_0^{X} e^{-x} a_{\alpha_0}(x)\, dx \right| + \left| \int_X^{\infty} e^{-x} a_\alpha(x)\, dx \right|$$
$$+ \left| \int_X^{\infty} e^{-x} a_{\alpha_0}(x)\, dx \right| < \epsilon,$$
if X is so chosen that each of the last two integrals is in

absolute value $< \frac{\epsilon}{3}$ for $x \geqq X$ and α is taken so close to α_0 that
$$\left| \int_0^{X} e^{-x} a_\alpha(x)\, dx - \int_0^{X} e^{-x} a_{\alpha_0}(x)\, dx \right| < \frac{\epsilon}{3}.$$
The last is a well-known property of integrals between finite limits, namely that they are continuous in a parameter in which their integrands are continuous. With this assumption the theorem is complete.

§ 13. Power series.

Perhaps the most interesting application of Borel's second definition of summation is in the theory of power series. We shall give three fundamental theorems in this theory, in spite of the fact that the proofs which we give use more of the theory of functions of a complex variable than has previously been the case.

The method of summation of the section is at all times that of Definition 50.

Let there be given a power series with finite radius of convergence, $R > 0$,

(1) $$a_0 + a_1 z + a_2 z^2 + \dots = \phi(z).$$

Suppose the function $\phi(z)$ thus defined to be analytically continued until it is defined as an analytic function over as much of the complex plane as is possible.

If Definition 50 will sum (1) over a continuously two-dimensional region of which the circle of convergence is a part, this sum function will be $\phi(z)$.

Theorem 228. HYPOTHESIS: (1) *is absolutely summable at M.* CONCLUSION: (1) *is absolutely summable over the line segment OM drawn from the origin to M.*

PROOF: Instead of the $a(x)$ of Definition 50 we write
$$a(\alpha, z) = a_0 + a_1 \frac{z\alpha}{1} + a_2 \frac{z^2\alpha^2}{2!} + a_3 \frac{z^3\alpha^3}{3!} + \dots.$$
We notice that this is a function of the product $z\alpha$. Consequently we write
$$a(\alpha, z) = F(\alpha z).$$

We usually shall denote differentiation by superscript.

Let $M = z_0 = \rho_0 e^{i\theta_0}$ and let a point on OM, neither O nor M, be $z = \rho e^{i\theta_0}$. We wish to prove

$$(2) \qquad \int_0^\infty e^{-\alpha} a(\alpha, z)\, d\alpha \qquad \text{and}$$

$$(3) \qquad \int_0^\infty e^{-\alpha} \left| \frac{d^\lambda}{d\alpha^\lambda} a(\alpha, z) \right| d\alpha, \quad \lambda = 0, 1, 2, \ldots$$

existent. We shall treat the second integral first.

$$\frac{d^\lambda}{d\alpha^\lambda} a(\alpha, z) = z^\lambda F^{(\lambda)}(\alpha z)\, d\alpha.$$

Make this substitution and drop from consideration the z^λ as being of no interest to the discussion. We have

$$(4) \qquad \int_0^\infty e^{-\alpha} | F^{(\lambda)}(\alpha z_0) | \, d\alpha$$

$$= \int_0^\infty e^{-\alpha} | F^{(\lambda)}(\alpha \rho_0 e^{i\theta_0}) | \, d\alpha$$

$$= \frac{1}{\rho_0} \int_0^\infty e^{-\frac{\beta}{\rho_0}} | F^{(\lambda)}(\beta e^{i\theta_0}) | \, d\beta.$$

The path of integration remains the axis of reals. At a fixed point z on OM we have

$$\int_0^\infty e^{-\alpha} | F^{(\lambda)}(\alpha z) | \, d\alpha = \frac{1}{\rho} \int_0^\infty e^{-\frac{\beta}{\rho}} | F^{(\lambda)}(\beta e^{i\theta_0}) | \, d\beta.$$

This is the integral of a real positive function; and since $0 < \rho < \rho_0$, it converges by comparison with (4).

We now examine (2).

$$\int_0^\infty e^{-\alpha} F(\alpha z_0)\, d\alpha = \frac{1}{\rho_0} \int_0^\infty e^{-\frac{\beta}{\rho_0}} F(\beta e^{i\theta_0})\, d\beta$$

converges. Multiply the integrand by the factor

$$\frac{\rho_0}{\rho} e^{\left(\frac{\beta}{\rho_0} - \frac{\beta}{\rho}\right)}$$

which is positive and monotonically approaches zero when $\beta \to \infty$. We still have a convergent integral *; that is,

* Readily proved by integration by parts.

$$\frac{1}{\rho} \int_0^\infty e^{-\frac{\beta}{\rho}} F(\beta e^{i\theta_0})\, d\beta = \int_0^\infty e^{-\alpha} F(\alpha z)\, d\alpha$$

converges.

Hence according to the definition (1) is absolutely summable at z. It is also absolutely summable at the origin and our theorem is proved.

Theorem 229. *Retain the notation and hypotheses of the last theorem.* CONCLUSION: *$\phi(z)$ is analytic at all points which are interior to or on the circle, c, drawn on OM as diameter.*

PROOF: As in the last proof,

$$\int_0^\infty e^{-\alpha} F(\alpha \rho_0 e^{i\theta_0})\, d\alpha = \frac{1}{\rho_0} \int_0^\infty e^{-\frac{\beta}{\rho_0}} F(\beta e^{i\theta_0})\, d\beta.$$

Multiply the integrand by $\frac{\rho_0}{|z'|} e^{\left(\frac{\beta}{\rho_0} - \frac{\beta}{z'}\right)}$. The integral will continue to converge if $\left| e^{\left(\frac{\beta}{\rho_0} - \frac{\beta}{z'}\right)} \right| \to 0$ monotonically when $\beta \to \infty$. Consequently,

$$(5) \qquad \frac{1}{z'} \int_0^\infty e^{-\frac{\beta}{z'}} F(\beta e^{i\theta_0})\, d\beta$$

exists if $\qquad R\left(e^{-\frac{\beta}{z'}}\right) \leq R\left(e^{-\frac{\beta}{\rho_0}}\right).$

This is true when and only when $z' \neq 0$ lies in or on c. We now recall that ϕ is given by this formula along the line OM. The formula, however, represents an analytic function over all the circle except the origin, and hence can be identified with ϕ over this region. Hence ϕ is analytic at each point z', of the region. We know it to be analytic at the origin.

It is to be noticed that (5) is not the formula for the summation of (1) at z' and that we have proved nothing to the summability of (1) at z'.

Theorem 230. *If a line is drawn from the origin to each singular point of ϕ and that portion of the plane bounded by perpendiculars to these lines at the singular points and including the origin is denoted by S; then* (1) *is absolutely*

summable within S to a function analytic at each point within S. It is not absolutely summable at any point outside S.

S may be an open region. It is called the polygon of summability.

It is to be noticed that the theorem makes no mention of points on the boundary of S.

PROOF: Consider a point M. Let O be the origin and draw a circle, c, on OM as diameter. We shall show that, if (1) has no singular point within or on c, (1) is absolutely summable at M.

Let c' be a circle with the same centre as c but radius greater by $\epsilon > 0$ but so small that $\phi(x)$ has no singular point within or on c'. Let the line OM cut c' in O' and M', O' next to O and M' next to M.

By Cauchy's integral theorem* and Theorem 145,

$$a_n = \frac{1}{2\pi i}\int_{c'} \frac{\phi(x)\,dx}{x^{n+1}}, \quad n = 0, 1, 2, \ldots.$$

Moreover, $\phi(x)\left[\frac{1}{x} + \frac{\alpha z}{x^2} + \frac{\alpha^2 z^2}{1 . 2 . x^3} + \ldots\right]$

converges uniformly in x over the circumference c'. Hence;

$$\begin{aligned} a(\alpha, z) &= a_0 + \frac{a_1 z \alpha}{1} + \frac{a_2 z^2 \alpha^2}{2!} + \ldots \\ &= \frac{1}{2\pi i}\int_{c'} \phi(x)\left[\frac{1}{x} + \frac{\alpha z}{x^2} + \frac{\alpha^2 z^2}{1 . 2 . x^3} + \ldots\right] dx. \\ &= \frac{1}{2\pi i}\int_{c'} \frac{1}{x}\,\phi(x)\, e^{\frac{\alpha z}{x}}\, dx. \end{aligned}$$

From this,

$$e^{-\alpha} a(\alpha, z) = \frac{1}{2\pi i}\int_{c'} \frac{\phi(x)}{x} e^{\alpha\left(\frac{z}{x} - 1\right)} dx.$$

Choose K so that, so long as x is on c', $\left|\frac{\phi(x)}{x}\right| < K$. Then

$$|e^{-\alpha} a(\alpha, z)| < \frac{K}{2\pi}\int_{c'} e^{\alpha R\left(\frac{z}{x} - 1\right)} dx.$$

* See, for example, W. F. Osgood, *Lehrbuch der Funktionentheorie*, S. 284, ed. 1912.

Suppose z so chosen that $R\left(\frac{z}{x} - 1\right) > \epsilon$ and let r' be the radius of c'. Then

$$|e^{-\alpha} a(\alpha, z)| < K r' e^{-\alpha\epsilon}.$$

Hence $$\int_0^\infty e^{-\alpha} a(\alpha, z)\, d\alpha \text{ exists.}$$

In an analogous manner we can prove the existence of

$$\int_0^\infty e^{-\alpha} |a^{(\lambda)}(\alpha, z)|\, d\alpha, \quad \lambda = 0, 1, 2, \ldots.$$

Consequently, (1) is absolutely summable when

$$R\left(1 - \frac{z}{x}\right) > \epsilon.$$

But ϵ is arbitrarily small and hence (1) is absolutely summable at z if $R\left(\frac{z}{x}\right) < 1$ for all values of x on c'. This condition is satisfied for a particular x by all points on the same side of the line drawn through x perpendicular to Ox as the origin and by no others. If x assumes all positions on c' these lines envelope an ellipse with foci at O and M. Consequently (1) is absolutely summable at all points within this ellipse. If there were but one singular point and corresponding perpendicular, l, every point on the same side of l as the origin would lie in the ellipse for some circles c and c'. It follows immediately that (1) is absolutely summable within S.

To show that (1) is not absolutely summable outside S, that is at a point, M, which is separated from the origin by one of the lines, l, which bound S, we notice that if we draw the line OM and construct a circle on it as diameter, this circle will include the singular point of ϕ which determines the line l. This contradicts Theorem 229 and completes the proof of the present theorem.

§ 14. Riesz means.

We shall consider a series,

(1) $$\sum_{n=1}^{\infty} a_n(z),$$

and a sequence, $\lambda_j \neq 0$ such that

$$\left|\sum_{j=1}^{n} \frac{1}{\lambda_j}\right| \to \infty .$$

Let
$$\sigma_n = \sum_{j=1}^{n} \frac{1}{\lambda_j}.$$

Moreover let
$$s_n^{(0)}(z) = \sum_{n=1}^{n} a_n(z),$$

$$s_n^{(1)}(z) = \frac{1}{\sigma_n} \sum_{n=1}^{n} \frac{1}{\lambda_n} s_n^{(0)}(z),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(2)
$$s_n^{(k)}(z) = \frac{1}{\sigma_n} \sum_{n=1}^{n} \frac{1}{\lambda_n} s_n^{(k-1)}(z).$$

Definition 53. *We call $s_n^{(k)}(z)$ the Riesz mean of order k for series (1). If $\lim_{n\to\infty} s_n^{(k)}(z)$ exists and equals $s(z)$ we say that the series is Riesz summable of order k and has the sum $s(z)$.*

The term summable in this section will refer to Riesz summability.

The Hölder method is a special case of the Riesz method with $\lambda_n = 1$.

Theorem 231. HYPOTHESIS: *Series (1) is uniformly summable of order $k-1$ to $s(z)$ over a set, I.* CONCLUSION: *(1) is uniformly summable of order k over I to $s(z)$.*

PROOF: Let $s_n^{(k-1)}(z) = s(z) + \sigma_n^{(k-1)}(z)$. Substitute in (2) and we observe that it is sufficient to prove the theorem in the case that $s(z) \equiv 0$. We make this assumption.

Let $\epsilon > 0$ be given. Choose N so large that

$$|s_n^{(k-1)}(z)| < \tfrac{1}{2}\epsilon$$

when $n > N$ for all z's of I simultaneously. We write

$$s_n^{(k)}(z) = \frac{1}{\sigma_n} \sum_{n=1}^{N} \frac{1}{\lambda_n} s_n^{(k-1)}(z) + \frac{1}{\sigma_n} \sum_{n=N+1}^{n} \frac{1}{\lambda_n} s_n^{(k-1)}(z).$$

Choose M so that for all values of z and n

$$\left|\frac{1}{\lambda_n} s_n^{(k-1)}(z)\right| < M.$$

Then,
$$|s_n^{(k)}(z)| \leqq \frac{NM}{|\sigma_n|} + \tfrac{1}{2}\epsilon < \epsilon \text{ if } |\sigma_n| > \frac{2NM}{\epsilon}.$$

Corollary. *Definition 53 is regular.*

This is readily proved by mathematical induction.

Theorem 232. HYPOTHESIS: *$\sum_{n=1}^{\infty} a_n$ is summable to s.* CONCLUSION: *$\sum_{n=k}^{\infty} a_n$ is summable to $s - \sum_{n=1}^{k-1} a_n$.*

Proof is elementary and is omitted.

Riesz summability is especially applicable to Dirichlet series.

We now consider the Dirichlet series

(3)
$$\sum_{n=1}^{\infty} c_n e^{-\sigma_{n-1} z},$$

where λ_n is real and $\lambda_n < \lambda_{n+1} \to \infty$. This λ_n is to be the λ_n of Definition 53.

Theorem 233. HYPOTHESIS: *Series (3) is Riesz-summable of order k at $z = z_0 = x_0 + y_0 i$ and $\frac{\lambda_{n+1}}{\lambda_n} \to l$.* CONCLUSION: *(3) is uniformly summable of order k throughout the region, R, defined by $x \geqq x_0 + \delta$ and $|z| < M$.*

PROOF: As in the proof of Theorem 163, we assume, without loss of generality, that $z_0 = 0$. We shall likewise assume $\lambda_1 > 0$. If this is not the case but $\lambda_k > 0$, discard from consideration the first $k-1$ terms of the series and divide through by $e^{-\sigma_{k-1}}$.

The proof of the theorem involves some transformations brought about by summation by parts; the formula for which is,

(4)
$$\sum_{i=1}^{n} u(i)\,\Delta v(i) = u(i)\,v(i)\Big]_1^{n+1} - \sum_{i=1}^{n} v(i+1)\,\Delta u(i).$$

The significance of the Δ is $\Delta v(i) = v(i+1) - v(i)$. We carry out these transformations in detail for the cases that $k = 0$, 1, and 2, in order that the reader may readily understand what is being done.

Let $s_n^{(0)}(z) = \sum_{n=1}^{n} c_n e^{-\sigma_{n-1}z}$ and $b_n(z) = e^{-\sigma_{n-1}z}$. We then write $s_n^{(0)}(z) = \sum_{n=1}^{n} c_n b_n(z)$ and apply formula (4). We get

$$(5) \qquad s_n^{(0)}(z) = b_{n+1}(z)\, s_n^{(0)}(0) - \sum_{n=1}^{n} [\Delta b_n(z)]\, s_n^{(0)}(0).$$

Apply the operator $\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}$ to both members of (5). Sum by parts again. We get,

$$(6) \quad s_n^{(1)}(z) = b_{n+2}(z)\, s_n^{(1)}(0) - \frac{1}{\sigma_n}\sum_{n=1}^{n}[\Delta b_{n+1}(z)]\,\sigma_n s_n^{(1)}(0)$$

$$- \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+1}}{\lambda_n}[\Delta b_{n+1}(z)]\,\sigma_n s_n^{(1)}(0)$$

$$+ \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\sum_{n=1}^{n}[\Delta(\lambda_n \Delta b_n(z))]\,\sigma_n s_n^{(1)}(0).$$

To get a formula for $s_n^{(2)}(z)$ we apply the operator

$$\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}$$

to (6) and sum by parts precisely as we have just done. We get the following:

$$(7) \quad s_n^{(2)}(z) = b_{n+3}(z)\, s_n^{(2)}(0)$$

$$- \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\sum_{n=1}^{n}[\Delta b_{n+2}(z)]\,\sigma_n s_n^{(2)}(0)$$

$$- \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+1}}{\lambda_n}[\Delta b_{n+2}(z)]\,\sigma_{n+1} s_n^{(2)}(0)$$

$$+ \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\frac{1}{\sigma_n}\sum_{n=1}^{n}[\Delta(\lambda_n(\Delta b_{n+1}(z))\,\sigma_n)]\,\sigma_n s_n^{(2)}(0)$$

$$- \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+2}}{\lambda_n}[\Delta b_{n+2}(z)]\,\sigma_{n+1} s_n^{(2)}(0)$$

$$+ \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\frac{1}{\sigma_n}\sum_{n=1}^{n}[\Delta(\lambda_{n+1}(\Delta b_{n+1}(z))\,\sigma_{n+1})]\,\sigma_n s_n^{(2)}(0)$$

$$+ \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+1}}{\lambda_n}[\Delta(\lambda_{n+1}(\Delta b_{n+1}(z))\,\sigma_{n+1})]\,\sigma_n s_n^{(2)}(0)$$

$$- \frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\sum_{n=1}^{n}[\Delta(\lambda_n \Delta(\lambda_n \Delta b_n(z))\,\sigma_n)]\, s_n^{(2)}(0).$$

Notice that formula (5) expresses $s_n^{(0)}(z)$ in terms of $s_n^{(0)}(0)$, that (6) expresses $s_n^{(1)}(z)$ in terms of $s_n^{(1)}(0)$ and (7) $s_n^{(2)}(z)$ in terms of $s_n^{(2)}(0)$. We repeat the process that we have just gone through with, repeatedly applying the operator

$$\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}$$

and summing by parts. We thus express $s_n^{(k)}(z)$ in terms of $s_n^{(k)}(0)$. We call this formula (8), although we do not write it out.

We shall carry the work forward from this point for $s_n^{(1)}(z)$ as given by (6) leaving to the reader the task of assuring himself of its generality as applied to (8).

We first assume that $s_n^{(1)}(0) \to 0$. This is brought about, if necessary, by replacing $s_n^{(1)}(0)$ by $s + \rho_n^{(1)}(0)$. Now let $\epsilon > 0$ be given and let N be so large that $|s_n^{(1)}(0)| < \epsilon$ when $n > N$. Take $n > N$. It results that

$$|s_n^{(1)}(z)| \leq \frac{1}{\sigma_n}\Big|\sum_{n=1}^{N}[\Delta b_{n+1}(z)]\,\sigma_n s_n^{(1)}(0)$$

$$- \sum_{n=1}^{N}\frac{\lambda_{n+1}}{\lambda_n}[\Delta b_{n+1}(z)]\,\sigma_n s_n^{(1)}(0)$$

$$+ \sum_{n=1}^{N}\frac{1}{\lambda_n}\sum_{n=1}^{n}[\Delta \lambda_n \Delta b_n(z)]\,\sigma_n s_n^{(1)}(0)\Big|$$

$$+\epsilon\left\{|b_{n+2}(z)|+\frac{1}{\sigma_n}\sum_{n=1}^{n}|\Delta b_{n+1}(z)|\sigma_n\right.$$

$$+\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+1}}{\lambda_n}|\Delta b_{n+1}(z)|\sigma_n$$

$$\left.+\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\sum_{n=1}^{n}|\Delta\lambda_n\Delta b_n(z)|\sigma_n\right\}.$$

Since for a particular ϵ N is fixed and since $\sigma_n \to \infty$ the theorem will be proved if we prove that the expression in the brace is always less than some constant. Consider each summand, for example the last. By Taylor's formula

$$\Delta\lambda_n\Delta b_n(z)=A(\lambda_n)e^{-\sigma_n z}\left[\frac{z^2}{\lambda_n}+\epsilon_n(z)\right],$$

where $A(\lambda_n)\neq 0$ and $\lambda_n\epsilon_n(z)\to 0$ uniformly over R. Consequently if x is the real part of z, for sufficiently great values of n

$$\frac{|\Delta\lambda_n\Delta b_n(z)|}{|\Delta\lambda_n\Delta b_n(x)|}<M\,(\text{constant})$$

for all z's of R simultaneously. It results that the expression in the brace bears a ratio that remains finite to the same expression with z replaced by its real part x. In this the absolute value signs can be dropped if the proper algebraic sign is used each time. Consequently we can replace the expression in the brace by

$$M\left\{b_{n+2}(x)-\frac{1}{\sigma_n}\sum_{n=1}^{n}(\Delta b_{n+1}(x))\sigma_n\right.$$

$$-\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{\lambda_{n+1}}{\lambda_n}(\Delta b_{n+1}(x))\sigma_n$$

$$\left.+\frac{1}{\sigma_n}\sum_{n=1}^{n}\frac{1}{\lambda_n}\sum_{n=1}^{n}(\Delta\lambda_n\Delta b_n(x))\sigma_n\right\}.$$

The expression in the brace here is $s_n^{(1)}(x)$ for the series for which $s_n^{(1)}(0)\equiv 1$, that is $1+0+0+0+\ldots$. By Theorem 231 this remains finite which is what we wished to prove.

As a corollary to this theorem we have the following theorem.

Theorem 234. HYPOTHESIS: (3) *is Riesz-summable of order k at z_0 and* $\frac{\lambda_{n+1}}{\lambda_n}\to l$. CONCLUSION: (3) *is Riesz-summable of order k over the half-plane defined by* $x > R(z_0)$.

The theory of summability of Dirichlet series could now be carried forward in a manner analogous to the way in which the theory of convergence of Dirichlet series was developed in Chapter XII. This is not done although several specific theorems are suggested in the exercises at the end of this chapter.

Summation by Riesz means is frequently treated in a way quite different* from what has just been done. There is a different definition which we give below as Definition 54. After the definition and a brief discussion we pass on to the next section.

Definition 54. *Let λ_n be a sequence of positive real monotonically increasing numbers becoming infinite when $n\to\infty$. Let $C_\lambda(\tau)=\Sigma a_n e^{-\lambda_n z}$ where the summation is extended to all values of λ_n such that $\lambda_n<\tau$ and let*

$$C_\lambda^K(\omega)=\sum_{\lambda_n<\omega}(\omega-\lambda_n)^K a_n e^{-\lambda_n z}=K\int_0^\infty C_\lambda(\tau)(\omega-\tau)^{K-1}\,d\tau.$$

If $\omega^{-K}C_\lambda^K(\omega)\to s$ as $\omega\to\infty$ we define s as the sum of the series $\sum_{n=1}^{\infty}a_n e^{-\lambda_n z}$.

This definition is in a way a generalization of the Hölder (Cesàro) sum. To show this let ω be an integer, $\lambda_n = n$ and $K=1$

$$\omega^{-K}C_\lambda^K(\omega)=\omega^{-1}\sum_{n=1}^{\omega-1}(\omega-n)a_n e^{-nz}=\frac{s_1+s_2+\ldots+s_{\omega-1}}{\omega}.$$

* *Comptes Rendus*, t. 149, p. 910; also *The General Theory of Dirichlet Series*, by G. H. Hardy and M. Riesz, &c.

If K is an integer greater than unity we have

$$K\omega^{-K}\int_0^{\omega} C_\lambda(\tau)\,(\omega-\tau)^{K-1}\,d\tau$$
$$= K!\,\omega^{-K}\int_{\tau_K=0}^{\omega}\int_{\tau_{K-1}=0}^{\tau_K}\cdots\int_{\tau_1=0}^{\tau_2} C_\lambda(\tau_1)\,d\tau_1\ldots d\tau_K,$$

as is readily proved by integration by parts. Notice that when τ is an integer $C_\lambda(\tau) = s_\tau$ and the resemblance of this to

$$K!\,\omega^{-K}\sum_{n_K=0}^{\omega}\sum_{n_{K-1}=0}^{n_K}\cdots\sum_{n_1=0}^{n_2} s_n,$$

is apparent. This last is a form of Cesàro mean that is not infrequently given. Compare it with (7) of section 2. For a continuous variable τ the above form is not an unnatural generalization [illegible] is the passage from n to λ_n.

§ 15. Other methods of summation.

A generalization of Borel's integral definition is *

Definition 55. $s = \int_0^\infty e^{-\alpha}\,u_p(\alpha)\,d\alpha,$

p = fixed integer $\geqq 0$, where

$$u_p(\alpha) = (a_0 + \ldots + a_{p-1})$$
$$+ (a_p + \ldots + a_{2p-1})\,\alpha + (a_{2p} + \ldots + a_{3p-1})\frac{\alpha^2}{2!} + \ldots.$$

Definition† 56. $s = \int_0^\infty e^{-\alpha}\,U_p(\alpha)\,d\alpha,$

p = fixed integer $\geqq 1$, and

$$U_p(\alpha) = \sum_{n=0}^{\infty}\frac{a_n\alpha^{np}}{(np)!}.$$

Definition ‡ 57. $s = \lim_{n\to\infty} y_n$ *where*

$$y_n = \sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!\,(k-1)!}\cdot\frac{(r-1)^{n-k}}{r^{n-1}}\,a_k.$$

* Borel, *Leçons*, p. 129.
† Le Roy, *Ann. de la Fac. des Sci. de Toulouse*, 1902, p. 217.
‡ Hausdorff, *Math. Zeitschrift*, B. 9, S. 86.

Definition * 58.

$$s = \lim_{t\to 1}\sum_{n=1}^{\infty}\frac{\Gamma(\overline{n-1}\,t+1)}{(n-1)!}\,a_n,\quad 0 < t < 1.$$

Definition † 59. $s = \lim_{t\to t_0}\sum_{n=1}^{\infty} a_n e^{-\lambda_n t},$

where t varies on T a point set having a limiting point t_0 not belonging to the set. λ_n is a sequence of monotonically increasing positive real numbers becoming infinite.

Still other definitions of the sum of a series have been given and studied. An exhaustive study here is out of the question.

§ 16. Further general discussion.

Several of the definitions of sum that have been given can be included under the following definition.

Definition 60. *A series, $\sum_{n=0}^{\infty} a_n$, will be said to be summed to s by a method of infinite reference in case $s = \lim_{n\to\infty} t_n$ where*

$$t_n = \sum_{k=0}^{\infty} c_{n,k}\,s_k.$$

$c_{n,k}$ is a set of numbers characteristic of the method of summation in question.

Definition 61. *The series will be said to be summed to s by a method of finite reference in case $s = \lim_{n\to\infty} t_n$ where*

$$t_n = \sum_{k=0}^{\infty} c_{n,k}\,s_k,$$

$c_{n,k}$ as above but $c_{n,k} = 0$ when $k > n$ and $c_{n,n} \neq 0$.

It will be observed that the latter definition is a special case of the former. The definition is warranted, however, since it includes important methods [illegible] that of Cesàro. The term infinite reference can be justified as distinguishing the general case from the important special case.

* Le Roy, *Ann. de la Fac. des Sci. de Toulouse*.
† Hardy and Chapman, *Quart. Journ.*, vol. 42, p. 198.

Theorem 235. *A necessary and sufficient condition that a definition by finite reference be regular is*

$$(1)\qquad \lim_{n\to\infty} c_{n,k} = 0 \text{ for every } k,$$

$$(2)\qquad \lim_{n\to\infty} \sum_{k=0}^{n} c_{n,k} = 1,$$

$$(3)\qquad \sum_{k=0}^{n} |c_{n,k}| < M \text{ (independent of } n).$$

PROOF: We shall first prove the condition necessary.

Select a particular k. Consider that series for which $s_k = 1$, and $s_i = 0$ when $i \neq k$. This must be summed to the value zero. This means that t_n must approach zero. But

$$t_n = c_{n,0}\, s_0 + \ldots + c_{n,k}\, s_k + \ldots + c_{n,n}\, s_n = c_{n,k}.$$

Hence, $c_{n,k} \to 0$.

Next consider the series for which $s_n \equiv 1$. For this series

$$t_n = c_{n,0} + c_{n,1} + \ldots + c_{n,n},$$

and hence $\qquad (c_{n,0} + c_{n,1} + \ldots + c_{n,n}) \to 1.$

The third part of the condition is somewhat more difficult to arrive at. We assume that the contrary is the case, namely that for a certain set of $c_{i,j}$'s, which define a regular definition,

$$A_n = |c_{n,0}| + \ldots + |c_{n,n}| \to \infty.$$

We shall set up a sequence s_n, approaching 0, but nevertheless such that for a certain sequence of values of n, t_n, formed with this set of $c_{i,j}$'s, becomes infinite. Let $\alpha > 1$ be fixed. Choose n_1 so that $A_{n_1} > \alpha^2$. If $c_{n_1,i} \neq 0$ put

$$s_i = \frac{1}{\alpha}\frac{|c_{n_1,i}|}{c_{n_1,i}},\quad i = 0, \ldots, n_1,$$

and if $c_{n_1,i} = 0$ put $s_i = \dfrac{1}{\alpha}$. Then $t_{n_1} = \dfrac{1}{\alpha} A_{n_1} > \alpha$. Next choose $n_2 > n_1$ so that

$$|c_{n_2,0}| + \ldots + |c_{n_2,n_1}| < \alpha \text{ and } A_{n_2} > \alpha^4 + 2\alpha^2,$$

simultaneously. To do this choose, first, $\bar{n}$ so great that for

it and all greater values of n each term $|c_{n,j}| < \dfrac{\alpha}{n_1}$ whenever $j \leqq n_1$. Then as is possible by the hypothesis choose $n_2 > \bar{n}$ so that $A_{n_2} > \alpha^4 + 2\alpha^2$. Then when $i = n_1 + 1, \ldots, n_2$ put

$$s_i = \frac{1}{\alpha^2}\cdot\frac{|c_{n_2,i}|}{c_{n_2,i}}$$

if $c_{n_2,i} \neq 0$ and $s_i = \dfrac{1}{\alpha^2}$ when $c_{n_2,i} = 0$. Then

$$|t_{n_2}| = \left|\sum_{i=0}^{n_2} c_{n_2,i}\, s_i\right| \geqq \left|\sum_{i=n_1+1}^{n_2} c_{n_2,i}\, s_i\right| - \sum_{i=0}^{n_1} |c_{n_2,i}\, s_i|$$

$$= \left|\frac{1}{\alpha^2} A_{n_2} - \frac{1}{\alpha^2}\sum_{i=0}^{n_1} |c_{n_2,i}|\right| - \sum_{i=0}^{n_1} |c_{n_2,i}\, s_i| > \left(\alpha^2 + 2 - \frac{1}{\alpha}\right) - 1 > \alpha^2.$$

Similarly, choose n_3 so that when $m \leqq n_2$,

$$|c_{n_3,0}| + \ldots + |c_{n_3,m}| < \alpha$$

and $A_{n_3} > \alpha^6 + 3\alpha^3$ simultaneously. When $i = n_2 + 1, \ldots, n_3$ put $s_i = \dfrac{1}{\alpha^3}\dfrac{|c_{n_3,i}|}{c_{n_3,i}}$ if $c_{n_3,i} \neq 0$, and $s_i = \dfrac{1}{\alpha^3}$ in case $c_{n_3,i} = 0$. We get

$$|t_{n_3}| = \left|\sum_{n_2+1}^{n_3} c_{n_3,i}\, s_i + \sum_{n_1+1}^{n_2} c_{n_3,i}\, s_i + \sum_{0}^{n_1} c_{n_3,i}\, s_i\right|$$

$$\geqq \left|\sum_{n_2+1}^{n_3} c_{n_3,i}\, s_i\right| - \left|\sum_{n_1+1}^{n_2} c_{n_3,i}\, s_i\right| - \left|\sum_{0}^{n_1} c_{n_3,i}\, s_i\right|$$

$$\geqq \left|\frac{1}{\alpha^3} A_{n_3} - \frac{1}{\alpha^3}\sum_{0}^{n_2} |c_{n_3,i}|\right| - \frac{1}{\alpha^2}\cdot\alpha - \frac{1}{\alpha}\cdot\alpha$$

$$> \left(\alpha^3 + 3 - \frac{1}{\alpha^2}\right) - \frac{1}{\alpha} - 1 > \alpha^3.$$

Next choose n_4 so that

$$A_{n_4} > \alpha^8 + 4\alpha^4 \text{ and } |c_{n_4,0}| + \ldots + |c_{n_4,m}| < \alpha$$

when $m \leqq n_3$. Omitting details we get $|t_{n_4}| > \alpha^4$. Proceeding, also omitting details for the general case, we get $|t_n| > \alpha^n$ which becomes infinite. This contradicts the

regularity of the definition and the necessity of the third condition is established.

We now wish to prove the sufficiency of the conditions. We shall assume that $s_n \to 0$. This is warranted; for

$$\begin{aligned}&\lim_{n\to\infty}(c_{n,0}s_0+\ldots+c_{n,n}s_n)\\ &= s\lim_{\to}(c_{n,0}+\ldots+c_{n,n})+\lim_{n\to\infty}(c_{n,0}(s_0-s)+\ldots+c_{n,n}(s_n-s))\\ &= s+\lim_{n\to\infty}(c_{n,0}(s_0-s)+\ldots+c_{n,n}(s_n-s)).\end{aligned}$$

Replace $s-s_i$ by s_i. Next, choose $\epsilon > 0$ and N so that when $n > N$ $|s_n| < \epsilon$. Then

$$t_n = (c_{n,0}s_0+\ldots+c_{n,N}s_N)+\sum_{i=N+1}^{n} c_{n,i}s_i.$$

The expression in parentheses approaches zero. The next sum is in absolute value less than ϵM which is arbitrarily small. It results that $t_n \to 0$, finishing the proof.

If we write

$$t_n = \sum_{k=0}^{\infty} b_{n,k}a_k$$

the following theorem results as a corollary to the one just proved.

Theorem 236. *A necessary and sufficient condition that a definition by finite reference be regular is*

(1) $$\lim_{n\to\infty} b_{n,k} = 1.$$

(2) $$\sum_{i=0}^{n-1} |b_{n,i}-b_{n,i+1}| < M.$$

It should be noticed that in the proof of the preceding two theorems the condition $c_{n,n} \neq 0$ is not used.

Theorem 237. *A sufficient condition that a method of summation by infinite reference be regular is*

(1) $$\lim_{n\to\infty} c_{n,k} = 0 \textit{ for every } k.$$

(2) $$\lim_{n\to\infty}\sum_{k=0}^{\infty} c_{n,k} = 1.$$

(3) $$\sum_{k=0}^{\infty} |c_{n,k}| < M \textit{ (independent of } n).$$

The first two parts of the condition are necessary for the regularity of the definition.

Proof of this theorem is similar to portions of the proof of Theorem 235 and is not given.

EXERCISES

322–340. Examine the series whose general terms are given below for summability according to various definitions:

$$(-1)^{n-1},\ (-1)^{n-1}n,\ (-1)^{n-1}n^2,\ (-1)^{n-1}n^k,\ (-1)^{n-1}n^{-x},$$
$$n^{-x},\ 1,\ n,\ n^2,\ n^k,\ \frac{n(n-1)\ldots(n-k+1)}{k!}x^n,\ (-1)^{n-1}e^{nx},$$
$$\frac{1}{n}e^{nx},\ (-1)^{n-1}\frac{1}{n}e^{nx},\ 2^{-n}e^{(\log\log n)\,x},\ e^{-(\log\log n)\,x},\ (-1)^{n-1}e^{-n^2x},$$
$$\frac{1}{n}\frac{n!}{x(x+1)\ldots(x+n)},\ \frac{\log 2 . \log 3 \ldots \log n}{(x+\log 2)(x+\log 3)\ldots(x+\log n)}.$$

341. Prove the theorem: HYPOTHESES: (i) $a_n \geqq a_{n+1} \to 0$; (ii) $a_0+a_1+\ldots+a_n = b_n$. CONCLUSION: $b_0-b_1+b_2-b_3+\ldots$ is summable C_1 with sum $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n$.

342. Prove the theorem: HYPOTHESIS: $\sum_{n=0}^{\infty} a_n$ is summable, C_1, with sum s. CONCLUSION: The series

$$a_0+\sum_{n=1}^{\infty}\frac{a_1+2a_2+\ldots+na_n}{n(n+1)}$$

converges to the sum s.

343. Prove the theorem: HYPOTHESES: (i) $\sum_{n=1}^{\infty} a_n$ is summable, C_1; (ii) $\sum_{n=1}^{\infty} n|a_n|^2$ converges. CONCLUSION: $\sum_{n=1}^{\infty} a_n$ converges.

344. Examine each of the methods of § 12 for regularity.

345. Investigate the boundary-value condition for each of the methods of § 12.

346, 347. State and prove theorems analogous to Theorems 164 and 166 for Riesz summability of Dirichlet series.

348. Discuss Riesz summability for

$$\sum_{n=1}^{\infty} c_n \frac{n!}{z(z+1)\dots(z+n)}.$$

Take $\lambda_n = n$.

349. Discuss Riesz summability for

$$\sum_{n=1}^{\infty} c_n \frac{\lambda_1 \dots \lambda_n}{(z+\lambda_1)\dots(z+\lambda_n)}.$$

CHAPTER XVIII

ASYMPTOTIC SERIES

The chief problems in a study of asymptotic series are, (i) the development of the series from the function frequently given implicitly by a differential or difference equation, (ii) the study of properties of the function by means of its corresponding asymptotic series. Such problems are omitted or but briefly studied in the present book. Consistent with this policy only the rudiments of the theory of asymptotic series are given.

Definition 62. *A series,*

(1) $$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \quad a_0,\ a_1,\ a_2,\ \dots$$

being constants, is said to represent a function $f(z)$ asymptotically along a certain radius vector if*

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n} + \epsilon_n,$$

where $\epsilon_n z^n \to 0$ when $z \to \infty$ along the radius vector in question.

The existence of divergent series asymptotic to certain functions will be assumed.

We shall use the notation

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots,$$

and when we deal with two asymptotic series at the same time we shall universally assume that the asymptotic vectors are the same.

Theorem 238. HYPOTHESIS:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

and $$f(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots.$$

CONCLUSION: $a_0 = b_0,\ a_1 = b_1,\ \dots,\ a_n = b_n,\ \dots.$

* This definition is frequently generalized so that the radius vector is replaced by a sector or even the whole plane.

PROOF: From Definition 62

$$0 = a_0 - b_0 + \frac{a_1 - b_1}{z} + \ldots + \frac{a_n - b_n}{z^n} + \epsilon_n^{(1)} - \epsilon_n^{(2)}, \tag{2}$$

where $(\epsilon_n^{(1)} - \epsilon_n^{(2)}) z^n \to 0$ when $z \to \infty$ along the asymptotic radius vector. Let $z \to \infty$ in this way and we have $a_0 = b_0$. Now multiply (2) through by z and repeat, finally arriving at the result $a_n = b_n$, where n is any particular integer.

Theorem 239. HYPOTHESIS:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

and

$$g(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots.$$

CONCLUSION:

$$f(z) \pm g(z) \sim (a_0 \pm b_0) + \frac{a_1 \pm b_1}{z} + \frac{a_2 \pm b_2}{z^2} + \ldots.$$

Proof is so simple as to be omitted.

Theorem 238 shows that a given function cannot have more than one asymptotic development along a given radius vector. However, the same series may be the asymptotic development for more than one function.

$$e^{-z} \sim 0 + \frac{0}{z} + \frac{0}{z^2} + \ldots$$

along the positive axis of reals. Hence if

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

along the positive axis of reals,

$$f(z) + e^{-z} \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

also.

Theorem 240. HYPOTHESIS:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots,$$

$$g(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots.$$

CONCLUSION: $$f(z)\,g(z) \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots,$$

where $$c_n = \sum_{r=0}^{n} a_r b_{n-r}.$$

PROOF: $$f(z) = a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \epsilon_n = f_n + \epsilon_n,$$

$$g(z) = b_0 + \frac{b_1}{z} + \ldots + \frac{b_n}{z^n} + \eta_n = g_n + \eta_n,$$

$$f(z)\,.\,g(z) = f_n \,.\, g_n + \epsilon_n g_n + \eta_n f_n + \epsilon_n \eta_n$$

$$= c_0 + \frac{c_1}{z} + \ldots + \frac{c_n}{z^n} + \theta_n + \epsilon_n g_n + \eta_n f_n + \epsilon_n \eta_n$$

where θ_n is either identically zero or a polynomial in $\frac{1}{z}$ of degree not less than $(n+1)$. When $z \to \infty$, $\theta_n z^n \to 0$, also $\epsilon_n g_n z^n$ and $\eta_n f_n z^n$. The theorem follows.

Theorem 241. HYPOTHESIS:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots,$$

$$g(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots, \quad b_0 \neq 0.$$

CONCLUSION: $$\frac{f(z)}{g(z)} \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots,$$

where the coefficients, $c_0, c_1, c_2, \ldots$, are obtained by dividing one series formally by the other according to the usual rule for polynomials in algebra, that is they are determined by the equations

$$b_0 c_0 = a_0,$$

$$b_1 c_0 + b_0 c_1 = a_1,$$

$$b_2 c_0 + b_1 c_1 + b_0 c_2 = a_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

PROOF: $\dfrac{f(z)}{g(z)} = \dfrac{f_n}{g_n} + \theta_n$, where $\theta_n = \dfrac{\epsilon_n - \dfrac{f_n}{g_n}\eta_n}{g_n + \eta_n}$. It is, of course, assumed that z is so large that the denominators do not vanish. When $z \to \infty$, $\epsilon_n z^n \to 0$, $\eta_n z^n \to 0$, $f_n \to a_0$, and $g_n \to b_0$. Consequently $z^n \theta_n \to 0$. Moreover $\dfrac{f_n}{g_n} = \sigma_n + \dfrac{\omega_n}{g_n}$, where σ_n is a polynomial of degree not greater than n in $\dfrac{1}{z}$ and ω_n is either identically zero or a polynomial of degree at least $n+1$ in $\dfrac{1}{z}$. $\omega_n z^n \to 0$ when $z \to \infty$. Hence

$$\frac{f(z)}{g(z)} = \sigma_n + \delta_n,$$

where $\delta_n z^n \to 0$. σ_n is obtained by the usual division rule. The theorem follows.

Theorem 242. HYPOTHESIS:

$$f(z) \sim \frac{a_2}{z^2} + \frac{a_3}{z^3} + \frac{a_4}{z^4} + \ldots.$$

CONCLUSION:

$$\int_z^\infty f(z)\,dz \sim \frac{a_2}{z} + \frac{a_3}{2z^2} + \frac{a_4}{3z^3} + \ldots, \tag{3}$$

the integration being along the asymptotic radius vector.

PROOF:

$$\int_z^\infty f(z)\,dz = \int_z^\infty f_n(z)\,dz + \int_z^\infty \epsilon_n(z)\,dz.$$

The first integral in the right-hand member exists and is obtained by integrating term by term. It gives the first n terms in the right-hand member of (3). Consider the second. When, with z on the asymptotic radius vector, $|z|$ is so large that $|\epsilon_n(z) z^n| < 1$ and $|b| > |z|$, b being also on the radius vector,

$$\left|\int_z^b \epsilon_n(z)\,dz\right| \leqq \int_z^b \frac{1}{|z|^n}|dz|$$
$$= \frac{1}{n+1}\left[\frac{1}{|z|^{n+1}} - \frac{1}{|b|^{n+1}}\right] \to \frac{1}{n+1}\cdot\frac{1}{|z|^{n+1}}$$

when $|b| \to \infty$. Consequently $\int_z^\infty \epsilon_n(z)\,dz$ exists and

$$z^n \int_z^\infty \epsilon_n(z)\,dz \to 0$$

when $z \to \infty$. The theorem follows.

Theorem 243. HYPOTHESIS:

$$f(z) \sim \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \text{ and} \tag{4}$$

$$f'(z) \sim \frac{b_2}{z^2} + \frac{b_3}{z^3} + \ldots. \tag{5}$$

CONCLUSION: *each term in* (5) *is the derivative of the corresponding term in* (4).

PROOF: Apply the previous theorem to development (5).

$$-f(z) \sim \frac{b_2}{z} + \frac{b_3}{2z^2} + \ldots.$$

But by Theorem 238 a function admits but one asymptotic development along a given vector, hence

$$b_2 = -a_1, \quad \frac{b_3}{2} = -a_2, \quad \ldots.$$

A function may have an asymptotic development and its first derivative not have an asymptotic development, along the same asymptotic radius vector at least. An example illustrative of this fact is

$$f(x) = e^{-x}\sin e^x \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \ldots$$

along the positive real axis.

$$f'(x) = -e^{-x}\sin e^x + \cos e^x$$

which is not asymptotic to

$$0 + \frac{0}{x} + \frac{0}{x^2} + \ldots,$$

and no other asymptotic development is possible along this radius vector by the previous theorem.

Theorem 244. HYPOTHESES:

(i) $f(z) = a_0 + w(z)$; (ii) $w(z) \sim \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$;

$$\text{(iii)}\ F(a_0 + w) = F(a_0) + F'(a_0)\,w + \frac{F''(a_0)}{2!}w^2 + \ldots$$

$$+ \frac{F^{(n-1)}(a_0)}{(n-1)!}w^{n-1} + \frac{F^{(n)}(a_0) + \epsilon_n(w)}{n!}w^n = F_n(w) + \frac{\epsilon_n(w)}{n!}w^n,$$

where $\lim\limits_{w \to 0} \epsilon_n(w) = 0$. CONCLUSION:

$$F(f) \sim F(a_0) + \frac{p_1}{z} + \frac{p_2}{z^2} + \ldots + \frac{p_n}{z^n} + \ldots,$$

where $p_1, p_2, \ldots, p_n, \ldots$, *are the coefficients of the successive powers of* $\frac{1}{z}$ *obtained by substituting in* $F_n(w)$ *the asymptotic development from* (ii) *of the hypothesis.*

PROOF: By combining theorems 239 and 240 we have

$$F(a_0) + F'(a_0)\,w + \frac{F''(a_0)}{2!}w^2 + \ldots$$
$$+ \frac{F^{(n)}(a_0)}{n!}w^n \sim F(a_0) + \frac{p_1}{z} + \frac{p_2}{z^2} + \ldots.$$

Hence

$$F(a_0 + w) = F(a_0) + \frac{p_1}{z} + \frac{p_2}{z^2} + \ldots + \frac{p_n + \eta_n(z)}{z^n} + \frac{\epsilon_n(w)}{n!}w^n,$$

where $\lim\limits_{z \to \infty} \eta_n(z) = 0$. Moreover when $z \longrightarrow \infty$ $\epsilon_n(w)\,w^n z^n \longrightarrow 0$ The theorem follows by Definition 62.

EXERCISES

350. Give an example of a function and corresponding divergent asymptotic series.

351. If $w(z)$ is the function given for Exercise 350, find an asymptotic development for $\sin(\pi + w(z))$.

352. Give an example, not given in the text, of two functions that have the same asymptotic development along the same radius vector.

353. The series of Definition 62 are called asymptotic power series. Give an analogous definition for asymptotic Dirichlet series.

354. Prove at least one theorem relative to asymptotic Dirichlet series as defined under Exercise 353.

PRINTED IN GREAT BRITAIN AT THE UNIVERSITY PRESS, OXFORD
BY JOHN JOHNSON, PRINTER TO THE UNIVERSITY

ERRATA

PAGE 53, ll. 6, 7, *for* number *read* sum

PAGE 111, add to HYPOTHESES

$$\int_{z_0}^{z} u_u'(z)\,dz \textit{ exists and is differentiable.}$$

PAGE 124, l. 15, for *points of a certain discrete set* read *points of a set of measure zero* * and make corresponding changes in the footnote and on page 125.

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